PARCC Model Content Frameworks

Mathematics

Grades 3–11

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Partnership for Assessment of Readiness for College and Careers

PARCC

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Purpose for the Model Content Frameworks for Mathematics

As part of its proposal to the U.S. Department of Education, the Partnership for Assessment of Readiness for College and Careers (PARCC) committed to developing model content frameworks for mathematics to serve as a bridge between the Common Core State Standards and the PARCC assessments.[1]

The PARCC Model Content Frameworks were developed through a state-led process that included mathematics content experts in PARCC member states and members of the Common Core State Standards writing team. The Model Content Frameworks are voluntary resources offered by PARCC to help curriculum developers and teachers as they work to implement the standards in their states and districts. They are designed with the following purposes in mind:

- Supporting implementation of the Common Core State Standards, and
- Informing the development of item specifications and blueprints for the PARCC assessments in grades 3–8 and high school.

The Model Content Frameworks are intended to be dynamic and responsive to evidence and ongoing input. As such, PARCC hopes they will be used by educators for the remainder of the 2011–12 school year. In spring 2012, PARCC will again solicit feedback on the Model Content Frameworks, and a refined version will be issued to incorporate feedback as needed. In this way, the Model Content Frameworks can evolve to reflect the real-life experiences of educators and students.

Connections to the PARCC Assessment

The PARCC Assessment System will be designed to measure the knowledge, skills and understandings essential to achieving college and career readiness. In mathematics, this includes conceptual understanding, procedural skill and fluency, and application and problem solving, as defined by the standards. Each of these works in conjunction with the others to promote students' achievement in mathematics. To measure the full range of the standards, the assessments will include tasks that require students to connect mathematical content and mathematical practices.

The Model Content Frameworks for Mathematics reflect these priorities by providing detailed information about selected practice standards, fluencies, connections and content emphases. These emphases will be reflected in the PARCC Assessment System.

The Model Content Frameworks do not contain a suggested scope and sequence by quarter.[2] Rather, they provides examples of key content dependencies (where one concept ought to come before another), key instructional emphases, opportunities for in-depth work on key concepts and connections

[1] The Model Content Frameworks, from grade 3 through grade 11, align with the PARCC Assessment System for those grades. A companion document with model content frameworks for grades K–2 will be written in 2012.

[2] See http://tinyurl.com/PARCCletter62411 for more information.

to critical practices. These last two components, in particular, intend to support local and state efforts to deliver instruction that connects content and practices while achieving the standards' balance of conceptual understanding, procedural skill and fluency, and application.

Overall, the PARCC Assessment System will include a mix of items, including short- and extended-response items, performance-based tasks, and technology-enhanced items.[3] In mathematics, the assessment system will be designed to measure students' understanding of key big ideas indicated in the standards, with emphasis on both the content standards and the practice standards. Questions asked will measure student learning within and across various mathematical domains and practices. The questions will cover the full range of mathematics, including conceptual understanding, procedural fluency and the varieties of expertise described by the practice standards. Mathematical understanding, procedural skill and the ability to apply what one knows are equally important and can be assessed using mathematical tasks of sufficient richness, which PARCC will include in its assessment system.

It is critical that all students are able to demonstrate mastery of the skills and knowledge described in the standards. PARCC recognizes the importance of equity, access and fairness in its assessments and aligned materials. To help meet these goals, PARCC will work with its Accessibility, Accommodations and Fairness Technical Working Group — a group of national experts — throughout the development process to ensure that the learning experience of all students is aligned to the high expectations of the standards.

Structure of the Model Content Frameworks for Mathematics

The Model Content Framework for Mathematics for each grade is written with the expectation that students develop content knowledge, conceptual understanding and expertise with the Standards for Mathematical Practice. A detailed description of all features of the standards would be significantly lengthier and denser. For that reason, the analyses given here are intended to be valuable starting points. The Model Content Frameworks for Mathematics provide guidance for grades 3–8 and high school across a number of areas:

- *Examples of key advances from the previous grade;*
- *Fluency expectations or examples of culminating standards;*
- *Examples of major within-grade dependencies;*
- *Examples of opportunities for connections among standards, clusters or domains;*
- *Examples of opportunities for in-depth focus;*
- *Examples of opportunities for connecting mathematical content and mathematical practices; and*
- *Content emphases by cluster.*

[3] Additional PARCC procurement documents, including the Item Specifications, will provide significantly greater detail about item types and how they will elicit evidence of student mastery of the Common Core State Standards in Mathematics.

Descriptions of each element are provided in “Grade-by-Grade Standards Analyses Introduction” (pages 11–14).

Principles Regarding the Common Core State Standards for Mathematics

Focus and Coherence

The two major evidence-based principles on which the standards are based are ***focus*** and ***coherence.*** **Focus** is necessary so that students have sufficient time to think, practice and integrate new ideas into their growing knowledge structure. Focus is also a way to allow time for the kinds of rich classroom discussion and interaction that support the Standards for Mathematical Practice.

The second principle, **coherence,** arises from mathematical connections. Some of the connections in the standards knit topics together at a single grade level (such as area models and multiplication in grade 3). Most connections, however, play out across two or more grade levels to form a progression of increasing knowledge, skill or sophistication. The standards are woven of these progressions. Likewise, instruction at any given grade would benefit from being informed by a sense of the overall progression students are following across the grades.

Another set of connections is found between the content standards and the practice standards. These connections are absolutely essential to support the development of students’ broader mathematical understanding. To reflect the standards, the Model Content Frameworks emphasize that mathematics is not a checklist of fragments to be mastered, but that doing and using mathematics involves connecting content and practices.

Focus is critical to ensure that students learn the most important content completely, rather than succumb to an overly broad survey of content. Coherence is critical to ensure that students see mathematics as a logically progressing discipline, which has intricate connections among its various domains and requires a sustained practice to master. Focus shifts over time, as seen in the following:

- In grades K–5, the focus is on the addition, subtraction, multiplication and division of whole numbers, fractions and decimals, with a balance of concepts, skills and problem solving. Arithmetic is viewed as an important set of skills and also as a thinking subject that, done thoughtfully, prepares students for algebra. Measurement and geometry develop alongside number and operations and are tied specifically to arithmetic along the way.

- In middle school, multiplication and division develop into powerful forms of ratio and proportional reasoning. The properties of operations take on prominence as arithmetic matures into algebra. The theme of quantitative relationships also becomes explicit in grades 6–8, developing into the formal notion of a function by grade 8. Meanwhile, the foundations of high school deductive geometry are laid in the middle grades. Finally, the gradual development of data representations in grades K–5 leads to statistics in middle school: the study of shape, center and spread of data distributions; possible associations between two variables; and the use of sampling in making statistical decisions.

- In high school, algebra, functions, geometry and statistics develop with an emphasis on modeling. Students continue to take a thinking approach to algebra, learning to see and make use of structure in algebraic expressions of growing complexity. As this description suggests, mathematical content in all grades is best approached in the ways envisioned by the Standards for Mathematical Practice.

The standards focus on crucial material so that students can have more time to discuss, reflect upon and practice it. The standards treat mathematics as a coherent subject to promote the sense-making that fuels mastery. The principles of focus and coherence are the twin engines that must be carried forward in implementation efforts and substantiated in curricula and assessments.

Connecting Content and Practices

The Standards for Mathematical Content and Standards for Mathematical Practice are meant to be connected, as noted in the Common Core State Standards for Mathematics (page 8):

> *Designers of curricula, assessments, and professional development should all attend to the need to connect the mathematical practices to mathematical content in mathematics instruction.*

The word *connect* in this passage is important. Separating the practices from the content is not helpful and is not what the standards require. The practices do not exist in isolation; the vehicle for engaging in the practices is mathematical content.

The Standards for Mathematical Practice should be embedded in classroom instruction, discussions and activities. They describe the kind of mathematics teaching and learning to be fostered in the classroom. To promote such an environment, students should have opportunities to work on carefully designed standards-based mathematical tasks that can vary in difficulty, context and type. Carefully designed standards-based mathematical tasks will reveal students' content knowledge and elicit evidence of mathematical practices. Mathematical tasks are an important opportunity to connect content and practices. To be consistent with the standards as a whole, assessment as well as curriculum and classroom activities must include a balance of mathematical tasks that provide opportunities for students to develop the kinds of expertise described in the practices.[4]

Higher Expectations: Conceptual Understanding, Fluency and Application

The standards are a rigorous set of expectations. According to these standards, it is not enough for students to learn procedures by rote. Nor, on the other hand, is it enough for students to "understand the concepts" without being able to apply them to solve problems. Nor, finally, is it enough for students to learn the important procedures of mathematics without attaining skill and fluency in them.

[4] To align with strong instruction, PARCC assessments will include several types of tasks. The task types will allow for integration of the content and practice standards. Task types will include shorter items and longer, constructed items, which will vary in technical difficulty. While they are not part of this document, PARCC is currently developing prototype assessment tasks that will be made available in the near future.

<u>*Conceptual understanding:*</u> A number of individual content standards use the word *understand* in connection with important mathematical concepts. As the standards state (page 4),

> *There is a world of difference between a student who can summon a mnemonic device to expand a product such as (a + b)(x + y) and a student who can explain where the mnemonic comes from. The student who can explain the rule understands the mathematics, and may have a better chance to succeed at a less familiar task such as expanding (a + b + c)(x + y).*

Conceptual understanding will be assessed using both short tasks and performance-based tasks as part of PARCC's commitment to measure the full range of the standards.

<u>*Procedural skill and fluency:*</u> As the standards state (page 4), "conceptual understanding and procedural skill are equally important." Thus, at various grade levels, specific content standards use the word *fluently.* These standards will be assessed as part of PARCC's commitment to measure the full range of the standards.

Wherever the word *fluently* appears in a content standard, the word means *quickly and accurately*. It means more or less the same as when someone is said to be fluent in a foreign language. To be fluent is to flow: Fluent isn't halting, stumbling or reversing oneself. A key aspect of fluency in this sense is that it is not something that happens all at once in a single grade but requires attention to student understanding along the way. It is important to ensure that sufficient practice and extra support are provided at each grade to allow all students to meet the standards that call explicitly for fluency.

<u>*Application:*</u> One of the mathematical practices is Modeling (MP.4), which sets an expectation that students will "apply the mathematics they know to problems arising in everyday life, society and the workplace." Modeling is further developed as a conceptual category in high school, where it is explicitly linked to mathematical content standards using the star symbols (see pages 72 and 73 of *Common Core State Standards for Mathematics*). Furthermore, many individual content standards refer explicitly to real-world problems. The ability to apply mathematics will be assessed as part of PARCC's commitment to measure the full range of the standards.

Guidance Regarding the Use of Resources in Mathematics

In the early phases of implementation, it is wise to consider the degree to which existing materials align to the standards. This is often done via "cross-walking" exercises. Such exercises are sometimes approached simplistically as a process of topic-matching. However, it is critical to note that individual content standards are carefully crafted statements — they do not simply name topics. Coverage of topics is therefore not a guarantee of alignment, and coverage may even affect alignment negatively when it is wide and/or shallow. Cluster headings often unify the standards in the cluster by communicating their joint intent. Aligning to the standards requires taking into account the guidance to be gained from cluster headings, grade-level introductions, indicators of opportunities for modeling or use of an applied approach, and so forth. In the context of a multigrade progression, alignment also means treating the content in ways that take into account the previous stage of the progression and anticipate the next.

One the primary purposes of the Model Content Frameworks for Mathematics is to provide educators with guidance on the implementation of the Common Core State Standards, particularly with respect to the needs of states and districts as they develop, obtain or revise materials to meet the standards. Therefore, a number of important criteria are suggested for reviewing existing resources or for the development of additional curricular or instructional materials if needed. These are presented in the form of a list that could support "strongly agree" to "strongly disagree" responses in any given case:

- Materials help students meet the indicated Standards for Mathematical Content. Materials also equip teachers and students to develop the varieties of expertise described in the Standards for Mathematical Practice.

- Materials are mathematically correct.

- Materials are motivating to students. The beauty and applied power of the subject is evident. Materials are engaging for a diverse body of students. This engagement exists side by side with the practice and hard thinking that is often necessary for learning mathematics.

- Materials reflect the standards by connecting content and practices while demanding conceptual understanding, procedural skill and fluency, and application.

  Specific aspects of achieving this balance include:

  Balance of tasks and activities: Activities, tasks and problems for students exhibit balance along various dimensions. For example, some activities and tasks target procedural skill and fluency alone; others target conceptual understanding; others application; and still others skill, understanding and application in equal measure. Some exercises are brief practice exercises; others require longer chains of reasoning. Some are abstract; others are contextual. Well-chosen tasks demonstrate the importance of mathematics in daily living for students, including connecting to other areas of students' interest, such as population growth and history, data and sports, and financial decision making.

  Balance in how time is spent: There is time for whole-class or group discussion and debate, time for solitary problem solving and reflection, and time for thoughtful practice and routine skill building. Individual problem solving and explanation of mathematical thinking may be intertwined several times during a class.

  Common sense in achieving balance: Not every task, activity or workweek has to be balanced in these ways. It is reasonable to have phases of narrow intensity, during which tasks, activities and time are concentrated in a single mode.

- Materials draw the teacher's attention explicitly to nuances in the content being addressed and to specific opportunities for teachers to foster mathematical practices in the study of that content.
- Materials give teachers workable strategies for helping students who have special needs, such as students with disabilities, English language learners and gifted students.
- Materials give teachers strategies for involving students in reading, writing, speaking and listening as necessary to meet the mathematics standards — for example, to understand the meanings of specialized vocabulary, symbols, units and expressions to support students in attending to precision (MP.6) or to engage in mathematical discourse using both informal

language and precise language to convey ideas, communicate solutions and support arguments (MP.3).

Notice that "coverage" is not in the above list. Materials that are excellent but narrow in scope still have value; they can be combined with other like resources and supplemented as necessary. This is better than settling for a single mediocre resource that claims to cover all content.

Additional Resources

Members of the working group and writing team for the Common Core State Standards for Mathematics are developing some resources to inform the development of curriculum and instruction aligned to the standards.

PARCC Resources

In the future, PARCC intends to build additional supplementary materials to further illustrate implementation of the standards, which may include model instructional units and sample tasks. In general, these materials will likely focus on areas of the standards that are particularly new to educators to support transition efforts. For example, the reader will note that grade 8 of the Model Content Frameworks includes a key opportunity for such work to occur around linear equations, the geometry of lines and proportional reasoning. As the materials become available, they will be published for voluntary use at http://parcconline.org.

Progressions

The progressions are being developed by members of the Common Core State Standards working group and writing team through the University of Arizona's Institute for Mathematics and Education. Progressions are narratives of the standards that describe how student skill and understanding in a particular domain develop from grade to grade. One of the primary uses of the progressions is to give educators and curriculum developers information that can help them develop materials for instruction aligned to the standards. http://ime.math.arizona.edu/progressions/.

Illustrative Mathematics

Under the guidance of members of the working group as well as other national experts in mathematics and mathematics education, The Illustrative Mathematics Project will illustrate the range and types of mathematical work that students will experience in a faithful implementation of the Common Core State Standards and by publishing other tools that support implementation of the standards. http://illustrativemathematics.org.

Common Core Tools

Additional tools that continue to be developed are posted from time to time on http://commoncoretools.wordpress.com, a blog moderated by Dr. William McCallum, distinguished professor and head of mathematics at the University of Arizona and mathematics lead for the Common Core State Standards for Mathematics.

GRADE-BY-GRADE STANDARDS ANALYSES INTRODUCTION

The following pages provide insights into the standards for grades 3–8. The reader is advised to have a copy of *Common Core State Standards for Mathematics* available for use in conjunction with this document. The Model Content Frameworks paraphrase standards and in some cases refer to them by code only; readers will need to refer to the standards document for exact language.

Description of Components

For each grade, analysis is provided in several categories. Please note: The words *examples* and *opportunities* in the following categories emphasize that the analysis provided in each category is not exhaustive. For example, there are many opportunities to connect mathematical content and practices in every grade, there are many opportunities for in-depth focus in every grade, and so on. A comprehensive description of these features of the standards would be hundreds of pages long. ***The analyses given here should be thought of as valuable starting points.***

Examples of Key Advances from the Previous Grade

- Highlights some of the major grade-to-grade steps in the progression of increasing knowledge and skill detailed in the standards. Note that each key advance in mathematical content also corresponds to a widening scope of problems that students can solve. Examples of key advances are highlighted to stress the need to treat topics in ways that take into account where students have been in previous grades and where they will be going in subsequent grades.[5]

Fluency Expectations or Examples of Culminating Standards

- Highlights individual standards that set expectations for fluency or that represent culminating masteries. Fluency standards are highlighted to stress the need to provide sufficient supports and opportunities for practice to help students meet these expectations. Culminating standards are highlighted to help give a sense of where important progressions are headed.

- Fluency is not meant to come at the expense of understanding but is an outcome of a progression of learning and sufficient thoughtful practice. It is important to provide the conceptual building blocks that develop understanding in tandem with skill along the way to fluency; the roots of this conceptual understanding often extend one or more grades earlier in the standards than the grade when fluency is finally expected.

Examples of Major Within-Grade Dependencies

- Highlights cases in which a body of content within a given grade depends conceptually or logically upon another body of content within that same grade. Examples of within-grade dependencies are highlighted to stress the need to organize material coherently within each given grade. (Because of space limitations, only examples of large-scale dependencies are described here, but coherence is important for dependencies that exist at finer grain size as well.)

[5] See the *Progressions* documents for additional information about progressions in the standards, http://ime.math.arizona.edu/progressions/.

Examples of Opportunities for Connections among Standards, Clusters or Domains

- Highlights opportunities for connecting content in assessments, as well as in curriculum and instruction. Examples of connections are highlighted to stress the need to avoid approaching the standards as merely a checklist.

Examples of Opportunities for In-Depth Focus

- Highlights some individual standards that play an important role in the content at each grade. The indicated mathematics might be given an especially in-depth treatment, as measured, for example, by the type of assessment items; the number of days; the quality of classroom activities to support varied methods, reasoning and explanation; the amount of student practice; and the rigor of expectations for depth of understanding or mastery of skills.[6]

Examples of Opportunities for Connecting Mathematical Content and Mathematical Practices

- Provides some examples of how students may engage in the mathematical practices as they learn the mathematics of the grade.[7] These examples are provided to stress the need to connect content and practices, as required by the standards.

- In addition to the concrete examples provided in each grade, the following are some general comments about connecting content and practices:

  - Connecting content and practices happens in the context of ***working on problems.*** The very first Standard for Mathematical Practice is to make sense of problems and persevere in solving them (MP.1).

  - Particularly in grades K–8, making sense of problems (MP.1) involves the pervasive use of ***visual representations as tools*** (MP.5) for understanding and explaining computation and problem solving with precision (MP.2, 6). Problem solving and explaining often require looking for and making use of structure (MP.7) and sometimes involve looking for and expressing regularity in repeated reasoning (MP.8).

  - As the above point suggests, the Standards for Mathematical Practice interact and overlap with each other, and several may be used together in solving a given problem. ***They are not a checklist.***

Content Emphases by Cluster

- Describes content emphases in the standards at the cluster level for each grade. These are provided because curriculum, instruction and assessment at each grade must reflect the focus and emphasis of the standards.

Not all of the content in a given grade is emphasized equally in the standards. The list of content standards for each grade is not a flat, one-dimensional checklist; this is by design. There are sometimes strong differences of emphasis even within a single domain. Some clusters require greater emphasis than the others based on the depth of the ideas, the time that they take to master, and/or their importance to future mathematics or the demands of college and career readiness. In addition, an

[6] Note, however, that a standard can be individually important even though the indicated mathematics may require relatively little teaching time.
[7] See the *Progressions* documents for additional examples, http://ime.math.arizona.edu/progressions/.

intense focus on the most critical material at each grade allows depth in learning, which is carried out through the Standards for Mathematical Practice. Without such focus, attention to the practices would be difficult and unrealistic, as would best practices like formative assessment.

Therefore, to make relative emphases in the standards more transparent and useful, the Model Content Frameworks designate clusters as **Major**, **Additional** and **Supporting** for the grade in question. As discussed further in Appendix C, some clusters that are not major emphases in themselves are designed to *support* and strengthen areas of major emphasis, while other clusters that may not connect tightly or explicitly to the major work of the grade would fairly be called *additional*.

To say that some things have greater emphasis is not to say that anything in the standards can safely be neglected in instruction. Neglecting material will leave gaps in student skill and understanding and may leave students unprepared for the challenges of a later grade. All standards figure in a mathematical education and will therefore be eligible for inclusion on the PARCC assessment. The assessments will mirror the message that is communicated here: Major Clusters will be a majority of the assessment, Supporting Clusters will be assessed through their success at supporting the Major Clusters and Additional Clusters will be assessed as well. The assessments will strongly focus where the standards strongly focus.

In addition to identifying the Major, Additional and Supporting Clusters for each grade, suggestions are given in each grade for ways to connect the Supporting Clusters to the Major Clusters of the grade. Thus, rather than suggesting even inadvertently that some material not be taught, there is direct advice for teaching it in ways that foster greater focus and coherence.

Finally, the following are some recommendations for using the cluster-level emphases:

Do …

- Use the guidance to inform instructional decisions regarding time and other resources spent on clusters of varying degrees of emphasis.
- Allow the focus on the major work of the grade to open up the time and space to bring the Standards for Mathematical Practice to life in mathematics instruction through sense-making, reasoning, arguing and critiquing, modeling, etc.
- Evaluate instructional materials taking the cluster-level emphases into account. The major work of the grade must be presented with the highest possible quality; the supporting work of the grade should indeed support the major focus, not detract from it.
- Set priorities for other implementation efforts taking the emphases into account, such as staff development; new curriculum development; or revision of existing formative or summative testing at the state, district or school level.

Don't …

- Neglect any material in the standards. (Instead, use the information provided to connect Supporting Clusters to the other work of the grade.)

- Sort clusters from Major to Supporting, and then teach them in that order. To do so would strip the coherence of the mathematical ideas and miss the opportunity to enhance the major work of the grade with the supporting clusters.

- Use the cluster headings as a replacement for the standards. All features of the standards matter — from the practices to surrounding text to the particular wording of individual content standards. Guidance is given at the cluster level as a way to talk about the content with the necessary specificity yet without going so far into detail as to compromise the coherence of the standards.

Examples of Key Advances from Grade 2 to Grade 3

- Students in grade 3 begin to enlarge their concept of number by developing an understanding of fractions as numbers. This work will continue in grades 3–6, preparing the way for work with the complete rational number system in grades 6 and 7.
- Students in grades K–2 worked on number; place value; and addition and subtraction concepts, skills and problem solving. Beginning in grade 3, students will learn concepts, skills and problem solving for multiplication and division. This work will continue in grades 3, 4 and 5, preparing the way for work with ratios and proportions in grades 6 and 7.

Fluency Expectations or Examples of Culminating Standards

3.OA.7 Students fluently multiply and divide within 100. By the end of grade 3, they know all products of two one-digit numbers from memory.

3.NBT.2 Students fluently add and subtract within 1000 using strategies and algorithms based on place value, properties of operations, and/or the relationship between addition and subtraction. (Although 3.OA.7 and 3.NBT.2 are both fluency standards, these two standards do not represent equal investments of time in grade 3. Note that students in grade 2 were already adding and subtracting within 1000, just not fluently. That makes 3.NBT.2 a relatively small and incremental expectation. By contrast, multiplication and division are new in grade 3, and meeting the multiplication and division fluency standard 3.OA.7 with understanding is a major portion of students' work in grade 3.)

Examples of Major Within-Grade Dependencies

- Students must begin work with multiplication and division (3.OA) at or near the very start of the year to allow time for understanding and fluency to develop. Note that area models for products are an important part of this process (3.MD.7). Hence, work on concepts of area (3.MD.5–6) should likely begin at or near the start of the year as well.

Examples of Opportunities for Connections among Standards, Clusters or Domains

- Students' work with partitioning shapes (3.G.2) relates to visual fraction models (3.NF).
- Scaled picture graphs and scaled bar graphs (3.MD.3) can be a visually appealing context for solving multiplication and division problems.

Examples of Opportunities for In-Depth Focus

3.OA.3 Word problems involving equal groups, arrays and measurement quantities can be used to build students' understanding of and skill with multiplication and division, as well as to allow students to demonstrate their understanding of and skill with these operations.

3.OA.7 Finding single-digit products and related quotients is a required fluency for grade 3. Reaching fluency will take much of the year for many students. These skills and the understandings that support them are crucial; students will rely on them for years to come as they learn to multiply and divide with multidigit whole numbers and to add, subtract, multiply and divide with fractions. After multiplication and division situations have been established, reasoning about patterns in products (e.g., products involving factors of 5 or 9) can help students remember particular products and quotients. Practice — and if necessary, extra support — should continue all year for those who need it to attain fluency.

3.NF.2 Developing an understanding of fractions as numbers is essential for future work with the number system. It is critical that students at this grade are able to place fractions on a number line diagram and understand them as a related component of their ever-expanding number system.

3.MD.2 Continuous measurement quantities such as liquid volume, mass and so on are an important context for fraction arithmetic (cf. 4.NF.4c, 5.NF.7c, 5.NF.3). In grade 3, students begin to get a feel for continuous measurement quantities and solve whole-number problems involving such quantities.

3.MD.7 Area is a major concept within measurement, and area models must function as a support for multiplicative reasoning in grade 3 and beyond.

Examples of Opportunities for Connecting Mathematical Content and Mathematical Practices

Mathematical practices should be evident *throughout* mathematics instruction and connected to all of the content areas highlighted above, as well as all other content areas addressed at this grade level. Mathematical tasks (short, long, scaffolded and unscaffolded) are an important opportunity to connect content and practices. Some brief examples of how the content of this grade might be connected to the practices follow.

- Students learn and use strategies for finding products and quotients that are based on the properties of operations; for example, to find 4×7, they may recognize that $7 = 5 + 2$ and compute $4 \times 5 + 4 \times 2$. This is an example of seeing and making use of structure (MP.7). Such reasoning processes amount to brief arguments that students may construct and critique (MP.3).

- Students will analyze a number of situation types for multiplication and division, including arrays and measurement contexts. Extending their understanding of multiplication and division to these situations requires that they make sense of problems and persevere in solving them

(MP.1), look for and make use of structure (MP.7) as they model these situations with mathematical forms (MP.4), and attend to precision (MP.6) as they distinguish different kinds of situations over time (MP.8).

Content Emphases by Cluster[8]

Not all of the content in a given grade is emphasized equally in the standards. Some clusters require greater emphasis than the others based on the depth of the ideas, the time that they take to master, and/or their importance to future mathematics or the demands of college and career readiness. In addition, an intense focus on the most critical material at each grade allows depth in learning, which is carried out through the Standards for Mathematical Practice.

To say that some things have greater emphasis is not to say that anything in the standards can safely be neglected in instruction. Neglecting material will leave gaps in student skill and understanding and may leave students unprepared for the challenges of a later grade. All standards figure in a mathematical education and will therefore be eligible for inclusion on the PARCC assessment. However, the assessments will strongly focus where the standards strongly focus.

In addition to identifying the Major, Additional and Supporting Clusters for each grade, suggestions are given following the table on the next page for ways to connect the Supporting to the Major Clusters of the grade. Thus, rather than suggesting even inadvertently that some material not be taught, there is direct advice for teaching it, in ways that foster greater focus and coherence.

[8] Refer to pages 12–14 for further explanation of the cluster-level emphases. Refer also to the Common Core State Standards for Mathematics for the standards that fall within each cluster.

Key: ■ Major Clusters; ◘ Supporting Clusters; ○ Additional Clusters

Operations and Algebraic Thinking

- ■ **Represent and solve problems involving multiplication and division.**
- ■ **Understand properties of multiplication and the relationship between multiplication and division.**
- ■ **Multiply and divide within 100.**
- ■ **Solve problems involving the four operations, and identify and explain patterns in arithmetic.**

Number and Operations in Base Ten

- ○ **Use place value understanding and properties of operations to perform multi-digit arithmetic.**

Number and Operations — Fractions

- ■ **Develop understanding of fractions as numbers.**

Measurement and Data

- ■ **Solve problems involving measurement and estimation of intervals of time, liquid volumes, and masses of objects.**
- ◘ **Represent and interpret data.**
- ■ **Geometric measurement: understand concepts of area and relate area to multiplication and addition.**
- ○ **Geometric measurement: recognize perimeter as an attribute of plane figures and distinguish between linear and area measures.**

Geometry

- ◘ **Reason with shapes and their attributes.**

Examples of Linking Supporting Clusters to the Major Work of the Grade

- Represent and interpret data: Students multiply and divide to solve problems using information presented in scaled bar graphs (3.MD.3). Pictographs and scaled bar graphs are a visually appealing context for one- and two-step word problems.

- Reason with shapes and their attributes: Work toward meeting 3.G.2 should be positioned in support of area measurement and understanding of fractions.

Examples of Key Advances from Grade 3 to Grade 4

- In grade 3, students studied multiplication in terms of equal groups, arrays and area. In grade 4, students extend their concept of multiplication to make multiplicative comparisons (4.OA.1).[9]

- Students in grade 4 apply and extend their understanding of the meanings and properties of addition and subtraction of whole numbers to extend addition and subtraction to fractions (4.NF.3).[10]

- Fraction equivalence is an important theme within the standards that begins in grade 3. In grade 4, students extend their understanding of fraction equivalence to the general case, $a/b = (n \times a)/(n \times b)$ (3.NF.3 $\rightarrow$ 4.NF.1).[11] They apply this understanding to compare fractions in the general case (3.NF.3d $\rightarrow$ 4.NF.2).

- Students in grade 4 apply and extend their understanding of the meanings and properties of multiplication of whole numbers to multiply a fraction by a whole number (4.NF.4).

- Students in grade 4 begin using the four operations to solve word problems involving measurement quantities such as liquid volume, mass and time (4.MD.2).

- Students combine their understanding of the meanings and properties of multiplication and division with their understanding of base-ten units to begin to multiply and divide multidigit numbers (4.NBT.5–6; this builds on work done in grade 3, cf. 3.NBT.3).

- Students generalize their previous understanding of place value for multidigit whole numbers (4.NBT.1–3). This supports their work in multidigit multiplication and division, carrying forward into grade 5, when students will extend place value to decimals.

Fluency Expectations or Examples of Culminating Standards

4.NBT.4 Students fluently add and subtract multidigit whole numbers using the standard algorithm.

[9] In an additive comparison problem (grades 1–2), the underlying question is *what amount would be added to one quantity to result in the other?* In a multiplicative comparison problem, the underlying question is *what factor would multiply one quantity to result in the other?*

[10] This work is limited to equal denominators in grade 4 to give students more time to build their understanding of fraction equivalence, before adding and subtracting unlike denominators in grade 5.

[11] Students who can generate equivalent fractions can also develop strategies for adding fractions with different denominators, but this is not a requirement in grade 4.

Examples of Major Within-Grade Dependencies

- Students' work with decimals (4.NF.5–7) depends to some extent on concepts of fraction equivalence and elements of fraction arithmetic. Students express fractions with a denominator of 10 as an equivalent fraction with a denominator of 100; comparisons of decimals require that students use similar reasoning to comparisons with fractions.

- Standard 4.MD.2 refers to using the four operations to solve word problems involving measurement quantities such as liquid volume, mass, time and so on. Some parts of this standard could be met earlier in the year (such as using whole-number multiplication to express measurements given in a larger unit in terms of a smaller unit — see also 4.MD.1), while others might be met only by the end of the year (such as word problems involving addition and subtraction of fractions or multiplication of a fraction by a whole number — see also 4.NF.3d and 4.NF.4c).

- Standard 4.MD.7 refers to word problems involving unknown angle measures. Before this standard can be met, students must understand concepts of angle measure (4.MD.5) and, presumably, gain some experience measuring angles (4.MD.6). Before that can happen, students must have some familiarity with the geometric terms that are used to define angles as geometric shapes (4.G.1).

Examples of Opportunities for Connections among Standards, Clusters or Domains

- The work that students do with units of measure (4.MD.1–2) and with multiplication of a fraction by a whole number (4.NF.4) can be connected to the idea of "times as much" in multiplication (4.OA.1).

- Addition of fractions (4.NF.3) and concepts of angle measure (4.MD.5a and 4.MD.7) are connected in that a one-degree measure is a fraction of an entire rotation and that adding angle measures together is adding fractions with a denominator of 360.

Examples of Opportunities for In-Depth Focus

4.NBT.5 When students work toward meeting this standard, they combine prior understanding of multiplication with deepening understanding of the base-ten system of units to express the product of two multidigit numbers as another multidigit number. This work will continue in grade 5 and culminate in fluency with the standard algorithms in grade 6.

4.NBT.6 When students work toward meeting this standard, they combine prior understanding of multiplication and division with deepening understanding of the base-ten system of units to find whole-number quotients and remainders with up to four-digit dividends and one-digit divisors. This work will develop further in grade 5 and culminate in fluency with the standard algorithms in grade 6.

4.NF.1 Extending fraction equivalence to the general case is necessary to extend arithmetic from whole numbers to fractions and decimals.

4.NF.3 This standard represents an important step in the multigrade progression for addition and subtraction of fractions. Students extend their prior understanding of addition and subtraction to add and subtract fractions with like denominators by thinking of adding or subtracting so many unit fractions.

4.NF.4 This standard represents an important step in the multigrade progression for multiplication and division of fractions. Students extend their developing understanding of multiplication to multiply a fraction by a whole number.

Examples of Opportunities for Connecting Mathematical Content and Mathematical Practices

Mathematical practices should be evident *throughout* mathematics instruction and connected to all of the content areas highlighted above, as well as all other content areas addressed at this grade level. Mathematical tasks (short, long, scaffolded and unscaffolded) are an important opportunity to connect content and practices. Some brief examples of how the content of this grade might be connected to the practices follow.

- When students decompose numbers into sums of multiples of base-ten units to multiply them (4.NBT.5), they are seeing and making use of structure (MP.7). As they illustrate and explain the calculation by using physical or drawn models, they are modeling (MP.4), using appropriate drawn tools strategically (MP.5) and attending to precision (MP.6) as they use base-ten units in the appropriate places.
- To compute and interpret remainders in word problems (4.OA.3), students must reason abstractly and quantitatively (MP.2), make sense of problems (MP.1), and look for and express regularity in repeated reasoning (MP.8) as they search for the structure (MP.7) in problems with similar interpretations of remainders.

Content Emphases by Cluster[12]

Not all of the content in a given grade is emphasized equally in the standards. Some clusters require greater emphasis than the others based on the depth of the ideas, the time that they take to master, and/or their importance to future mathematics or the demands of college and career readiness. In addition, an intense focus on the most critical material at each grade allows depth in learning, which is carried out through the Standards for Mathematical Practice.

To say that some things have greater emphasis is not to say that anything in the standards can safely be neglected in instruction. Neglecting material will leave gaps in student skill and understanding and may leave students unprepared for the challenges of a later grade. All standards figure in a mathematical education and will therefore be eligible for inclusion on the PARCC assessment. However, the assessments will strongly focus where the standards strongly focus.

[12] Refer to pages 12–14 for further explanation of the cluster-level emphases. Refer also to the Common Core State Standards for Mathematics for the standards that fall within each cluster.

In addition to identifying the Major, Additional and Supporting Clusters for each grade, suggestions are given following the table for ways to connect the Supporting to the Major Clusters of the grade. Thus, rather than suggesting even inadvertently that some material not be taught, there is direct advice for teaching it, in ways that foster greater focus and coherence.

Key: ■ Major Clusters; ◘ Supporting Clusters; ○ Additional Clusters

Operations and Algebraic Thinking	
■	**Use the four operations with whole numbers to solve problems.**
◘	**Gain familiarity with factors and multiples.**
○	**Generate and analyze patterns.**
Number and Operations in Base Ten	
■	**Generalize place value understanding for multi-digit whole numbers.**
■	**Use place value understanding and properties of operations to perform multi-digit arithmetic.**
Number and Operations — Fractions	
■	**Extend understanding of fraction equivalence and ordering.**
■	**Build fractions from unit fractions by applying and extending previous understandings of operations on whole numbers.**
■	**Understand decimal notation for fractions, and compare decimal fractions.**
Measurement and Data	
◘	**Solve problems involving measurement and conversion of measurements from a larger unit to a smaller unit.**
◘	**Represent and interpret data.**
○	**Geometric measurement: understand concepts of angle and measure angles.**
Geometry	
○	**Draw and identify lines and angles, and classify shapes by properties of their lines and angles.**

Examples of Linking Supporting Clusters to the Major Work of the Grade

- Gain familiarity with factors and multiples: Work in this cluster supports students' work with multidigit arithmetic as well as their work with fraction equivalence.
- Represent and interpret data: The standard in this cluster requires students to use a line plot to display measurements in fractions of a unit and to solve problems involving addition and subtraction of fractions, connecting it directly to the Number and Operations — Fractions clusters.

Examples of Key Advances from Grade 4 to Grade 5

- In grade 5, students will integrate decimal fractions more fully into the place value system (5.NBT.1–4). By thinking about decimals as sums of multiples of base-ten units, students begin to extend algorithms for multidigit operations to decimals (5.NBT.7).
- Students use their understanding of fraction equivalence and their skill in generating equivalent fractions as a strategy to add and subtract fractions, including fractions with unlike denominators.
- Students apply and extend their previous understanding of multiplication to multiply a fraction or whole number by a fraction (5.NF.4). They also learn the relationship between fractions and division, allowing them to divide any whole number by any nonzero whole number and express the answer in the form of a fraction or mixed number (5.NF.3). And they apply and extend their previous understanding of multiplication and division to divide a unit fraction by a whole number or a whole number by a unit fraction.[13]
- Students extend their grade 4 work in finding whole-number quotients and remainders to the case of two-digit divisors (5.NBT.6).
- Students continue their work in geometric measurement by working with volume as an attribute of solid figures and as a measurement quantity (5.MD.3–5).
- Students build on their previous work with number lines to use two perpendicular number lines to define a coordinate system (5.G.1–2).

Fluency Expectations or Examples of Culminating Standards

5.NBT.5 Students fluently multiply multidigit whole numbers using the standard algorithm.

Examples of Major Within-Grade Dependencies

- Understanding that in a multidigit number, a digit in one place represents $1/10$ of what it represents in the place to its left (5.NBT.1) is an example of multiplying a quantity by a fraction (5.NF.4).

[13] Students able to multiply fractions in general can develop strategies to divide fractions in general by reasoning about the relationship between multiplication and division. But the division of a fraction by a fraction is not a requirement in this grade.

Examples of Opportunities for Connections among Standards, Clusters or Domains

- The work that students do in multiplying fractions extends their understanding of the operation of multiplication. For example, to multiply $a/b \times q$ (where q is a whole number or a fraction), students can interpret $a/b \times q$ as meaning a parts of a partition of q into b equal parts (5.NF.4a). This interpretation of the product leads to a product that is less than, equal to or greater than q depending on whether $a/b < 1$, $a/b = 1$ or $a/b > 1$, respectively (5.NF.5).

- Conversions within the metric system represent an important practical application of the place value system. Students' work with these units (5.MD.1) can be connected to their work with place value (5.NBT.1).

Examples of Opportunities for In-Depth Focus

5.NBT.1 The extension of the place value system from whole numbers to decimals is a major intellectual accomplishment involving understanding and skill with base-ten units and fractions.

5.NBT.6 The extension from one-digit divisors to two-digit divisors requires care. This is a major milestone along the way to reaching fluency with the standard algorithm in grade 6 (6.NS.2).

5.NF.2 When students meet this standard, they bring together the threads of fraction equivalence (grades 3–5) and addition and subtraction (grades K–4) to fully extend addition and subtraction to fractions.

5.NF.4 When students meet this standard, they fully extend multiplication to fractions, making division of fractions in grade 6 (6.NS.1) a near target.

5.MD.5 Students work with volume as an attribute of a solid figure and as a measurement quantity. Students also relate volume to multiplication and addition. This work begins a progression leading to valuable skills in geometric measurement in middle school.

Examples of Opportunities for Connecting Mathematical Content and Mathematical Practices

Mathematical practices should be evident *throughout* mathematics instruction and connected to all of the content areas highlighted above, as well as all other content areas addressed at this grade level. Mathematical tasks (short, long, scaffolded and unscaffolded) are an important opportunity to connect content and practices. Some brief examples of how the content of this grade might be connected to the practices follow.

- When students break divisors and dividends into sums of multiples of base-ten units (5.NBT.6), they are seeing and making use of structure (MP.7) and attending to precision (MP.6). Initially for most students, multidigit division problems take time and effort, so they also require perseverance (MP.1) and looking for and expressing regularity in repeated reasoning (MP.8).

- When students explain patterns in the number of zeros of the product when multiplying a number by powers of 10 (5.NBT.2), they have an opportunity to look for and express regularity in repeated reasoning (MP.8). When they use these patterns in division, they are making sense of problems (MP.1) and reasoning abstractly and quantitatively (MP.2).

- When students show that the volume of a right rectangular prism is the same as would be found by multiplying the side lengths (5.MD.5), they also have an opportunity to look for and express regularity in repeated reasoning (MP.8). They attend to precision (MP.6) as they use correct length or volume units, and they use appropriate tools strategically (MP.5) as they understand or make drawings to show these units.

Content Emphases by Cluster[14]

Not all of the content in a given grade is emphasized equally in the standards. Some clusters require greater emphasis than the others based on the depth of the ideas, the time that they take to master, and/or their importance to future mathematics or the demands of college and career readiness. In addition, an intense focus on the most critical material at each grade allows depth in learning, which is carried out through the Standards for Mathematical Practice.

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In addition to identifying the Major, Additional and Supporting Clusters for each grade, suggestions are given following the table on the next page for ways to connect the Supporting to the Major Clusters of the grade. Thus, rather than suggesting even inadvertently that some material not be taught, there is direct advice for teaching it, in ways that foster greater focus and coherence.

[14] Refer to pages 12–14 for further explanation of the cluster-level emphases. Refer also to the Common Core State Standards for Mathematics for the standards that fall within each cluster.

Key: ■ Major Clusters; ◻ Supporting Clusters; ○ Additional Clusters

Operations and Algebraic Thinking

- ○ **Write and interpret numerical expressions.**
- ○ **Analyze patterns and relationships.**

Number and Operations in Base Ten

- ■ **Understand the place value system.**
- ■ **Perform operations with multi-digit whole numbers and with decimals to hundredths.**

Number and Operations — Fractions

- ■ **Use equivalent fractions as a strategy to add and subtract fractions.**
- ■ **Apply and extend previous understandings of multiplication and division to multiply and divide fractions.**

Measurement and Data

- ◻ **Convert like measurement units within a given measurement system.**
- ◻ **Represent and interpret data.**
- ■ **Geometric measurement: understand concepts of volume and relate volume to multiplication and to addition.**

Geometry

- ○ **Graph points on the coordinate plane to solve real-world and mathematical problems.**
- ○ **Classify two-dimensional figures into categories based on their properties.**

Examples of Linking Supporting Clusters to the Major Work of the Grade

- Convert like measurement units within a given measurement system: Work in these standards supports computation with decimals. For example, converting 5 cm to 0.05 m involves computation with decimals to hundredths.

- Represent and interpret data: The standard in this cluster provides an opportunity for solving real-world problems with operations on fractions, connecting directly to both Number and Operations — Fractions clusters.

Examples of Key Advances from Grade 5 to Grade 6

- Students' prior understanding of and skill with multiplication, division and fractions contribute to their study of ratios, proportional relationships and unit rates (6.RP).
- Students begin using properties of operations systematically to work with variables, variable expressions and equations (6.EE).
- Students extend their work with the system of rational numbers to include using positive and negative numbers to describe quantities (6.NS.5), extending the number line and coordinate plane to represent rational numbers and ordered pairs (6.NS.6), and understanding ordering and absolute value of rational numbers (6.NS.7).
- Having worked with measurement data in previous grades, students begin to develop notions of statistical variability, summarizing and describing distributions (6.SP).

Fluency Expectations or Examples of Culminating Standards

6.NS.2 Students fluently divide multidigit numbers using the standard algorithm. This is the culminating standard for several years' worth of work with division of whole numbers.

6.NS.3 Students fluently add, subtract, multiply and divide multidigit decimals using the standard algorithm for each operation. This is the culminating standard for several years' worth of work relating to the domains of Number and Operations in Base Ten, Operations and Algebraic Thinking, and Number and Operations — Fractions.

6.NS.1 Students interpret and compute quotients of fractions and solve word problems involving division of fractions by fractions. This completes the extension of operations to fractions.

Examples of Major Within-Grade Dependencies

- Equations of the form $px = q$ (6.EE.7) are unknown-factor problems; the solution will sometimes be the quotient of a fraction by a fraction (6.NS.1).
- Solving problems by writing and solving equations (6.EE.7) involves not only an appreciation of how variables are used (6.EE.6) and what it means to solve an equation (6.EE.5) but also some ability to write, read and evaluate expressions in which letters stand for numbers (6.EE.2).
- Students must be able to place rational numbers on a number line (6.NS.7) before they can place ordered pairs of rational numbers on a coordinate plane (6.NS.8). The former standard about ordering rational numbers is much more fundamental.

Examples of Opportunities for Connections among Standards, Clusters or Domains

- Students' work with ratios and proportional relationships (6.RP) can be combined with their work in representing quantitative relationships between dependent and independent variables (6.EE.9).

- Plotting rational numbers in the coordinate plane (6.NS.8) is part of analyzing proportional relationships (6.RP.3a, 7.RP.2) and will become important for studying linear equations (8.EE.8) and graphs of functions (8.F).[15]

- Students use their skill in recognizing common factors (6.NS.4) to rewrite expressions (6.EE.3).

- Writing, reading, evaluating and transforming variable expressions (6.EE.1–4) and solving equations and inequalities (6.EE.7–8) can be combined with use of the volume formulas $V = lwh$ and $V = Bh$ (6.G.2).

- Working with data sets can connect to estimation and mental computation. For example, in a situation where there are 20 different numbers that are all between 8 and 10, one might quickly estimate the sum of the numbers as $9 \times 20 = 180$.

Examples of Opportunities for In-Depth Focus

6.RP.3 When students work toward meeting this standard, they use a range of reasoning and representations to analyze proportional relationships.

6.NS.1 This is a culminating standard for extending multiplication and division to fractions.

6.NS.8 When students work with rational numbers in the coordinate plane to solve problems, they combine and consolidate elements from the other standards in this cluster.

6.EE.3 By applying properties of operations to generate equivalent expressions, students use properties of operations that they are familiar with from previous grades' work with numbers — generalizing arithmetic in the process.

6.EE.7 When students write equations of the form $x + p = q$ and $px = q$ to solve real-world and mathematical problems, they draw on meanings of operations that they are familiar with from previous grades' work. They also begin to learn algebraic approaches to solving problems.[16]

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[15] While not required by the standards, it might be considered valuable to expose students to time series data and to time graphs as an appealing way to work with rational numbers in the coordinate plane (6.NS.8). For example, students could create time graphs of temperature measured each hour over a 24-hour period in a place where, to ensure a strong connection to rational numbers, temperature values might cross from positive to negative during the night and back to positive the next day.

[16] For example, suppose Daniel went to visit his grandmother, who gave him \$5.50. Then he bought a book costing \$9.20 and had \$2.30 left. To find how much money he had before visiting his grandmother, an algebraic approach leads to the equation $x + 5.50 - 9.20 = 2.30$. An arithmetic approach without using variables at all would be to begin with 2.30, then add 9.20, then

Examples of Opportunities for Connecting Mathematical Content and Mathematical Practices

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- Reading and transforming expressions involves seeing and making use of structure (MP.7). Relating expressions to situations requires making sense of problems (MP.1) and reasoning abstractly and quantitatively (MP.2).
- The sequence of steps in the solution of an equation is a logical argument that students can construct and critique (MP.3). Such arguments require looking for and making use of structure (MP.7) and, over time, expressing regularity in repeated reasoning (MP.8).
- Thinking about the point (1, r) in a graph of a proportional relationship with unit rate r involves reasoning abstractly and quantitatively (MP.2). The graph models with mathematics (MP.4) and uses appropriate tools strategically (MP.5).
- Area, surface area and volume present modeling opportunities (MP.4) and require students to attend to precision with the types of units involved (MP.6).
- Students think with precision (MP.6) and reason quantitatively (MP.2) when they use variables to represent numbers and write expressions and equations to solve a problem (6.EE.6–7).
- Working with data gives students an opportunity to use appropriate tools strategically (MP.5). For example, spreadsheets can be powerful for working with a data set with dozens or hundreds of data points.

Content Emphases by Cluster[17]

Not all of the content in a given grade is emphasized equally in the standards. Some clusters require greater emphasis than the others based on the depth of the ideas, the time that they take to master, and/or their importance to future mathematics or the demands of college and career readiness. In addition, an intense focus on the most critical material at each grade allows depth in learning, which is carried out through the Standards for Mathematical Practice.

subtract 5.50. This yields the desired answer, but students will eventually encounter problems in which arithmetic approaches are unrealistically difficult and algebraic approaches must be used.

[17] Refer to pages 12–14 for further explanation of the cluster-level emphases. Refer also to the Common Core State Standards for Mathematics for the standards that fall within each cluster.

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In addition to identifying the Major, Additional and Supporting Clusters for each grade, suggestions are given following the table for ways to connect the Supporting to the Major Clusters of the grade. Thus, rather than suggesting even inadvertently that some material not be taught, there is direct advice for teaching it, in ways that foster greater focus and coherence.

Key: ■ Major Clusters; ▣ Supporting Clusters; ○ Additional Clusters

Ratios and Proportional Reasoning	
■	**Understand ratio concepts and use ratio reasoning to solve problems.**
The Number System	
■	**Apply and extend previous understandings of multiplication and division to divide fractions by fractions.**
○	**Compute fluently with multi-digit numbers and find common factors and multiples.**
■	**Apply and extend previous understandings of numbers to the system of rational numbers.**
Expressions and Equations	
■	**Apply and extend previous understandings of arithmetic to algebraic expressions.**
■	**Reason about and solve one-variable equations and inequalities.**
■	**Represent and analyze quantitative relationships between dependent and independent variables.**
Geometry	
▣	**Solve real-world and mathematical problems involving area, surface area and volume.**
Statistics and Probability	
○	**Develop understanding of statistical variability.**
○	**Summarize and describe distributions.**

Examples of Linking Supporting Clusters to the Major Work of the Grade

- Solve real-world and mathematical problems involving area, surface area and volume: In this cluster, students work on problems with areas of triangles and volumes of right rectangular prisms, which connects to work in the Expressions and Equations domain. In addition, another standard within this cluster asks students to draw polygons in the coordinate plane, which supports other work with the coordinate plane in The Number System domain.

Examples of Key Advances from Grade 6 to Grade 7

- In grade 6, students learned about negative numbers and the kinds of quantities they can be used to represent; they also learned about absolute value and ordering of rational numbers, including in real-world contexts. In grade 7, students will add, subtract, multiply and divide within the system of rational numbers.

- Students grow in their ability to analyze proportional relationships. They decide whether two quantities are in a proportional relationship (7.RP.2a); they work with percents, including simple interest, percent increase and decrease, tax, markups and markdowns, gratuities and commission, and percent error (7.RP.3); they analyze proportional relationships and solve problems involving unit rates associated with ratios of fractions (e.g., if a person walks $^1/_2$ mile in each $^1/_4$ hour, the unit rate is the complex fraction ½ / ¼ miles per hour or 2 miles per hour) (7.RP.1); and they analyze proportional relationships in geometric figures (7.G.1).

- Students solve a variety of problems involving angle measure, area, surface area and volume (7.G.4–6).

Fluency Expectations or Examples of Culminating Standards

7.EE.3 Students solve multistep problems posed with positive and negative rational numbers in any form (whole numbers, fractions and decimals), using tools strategically. This work is the culmination of many progressions of learning in arithmetic, problem solving and mathematical practices.

7.EE.4 In solving word problems leading to one-variable equations of the form $px + q = r$ and $p(x + q) = r$, students solve the equations fluently. This will require fluency with rational number arithmetic (7.NS.1–3), as well as fluency to some extent with applying properties operations to rewrite linear expressions with rational coefficients (7.EE.1).

7.NS.1–2 Adding, subtracting, multiplying and dividing rational numbers is the culmination of numerical work with the four basic operations. The number system will continue to develop in grade 8, expanding to become the real numbers by the introduction of irrational numbers, and will develop further in high school, expanding to become the complex numbers with the introduction of imaginary numbers. Because there are no specific standards for rational number arithmetic in later grades and because so much other work in grade 7 depends on rational number arithmetic (see below), fluency with rational number arithmetic should be the goal in grade 7.

Examples of Major Within-Grade Dependencies

- Meeting standard 7.EE.3 in its entirety will involve using rational number arithmetic (7.NS.1–3) and percents (7.RP.3). Work leading to meeting this standard could be organized as a recurring activity that tracks the students' ongoing acquisition of new skills in rational number arithmetic and percents.
- Because rational number arithmetic (7.NS.1–3) underlies the problem solving detailed in 7.EE.3 as well as the solution of linear expressions and equations (7.EE.1–2, 4), this work should likely begin at or near the start of the year.
- The work leading to meeting standards 7.EE.1–4 could be divided into two phases, one centered on addition and subtraction (e.g., solving $x + q = r$) in relation to rational number addition and subtraction (7.NS.1) and another centered on multiplication and division (e.g., solving $px + q = r$ and $p(x + q) = r$) in relation to rational number multiplication and division (7.NS.2).

Examples of Opportunities for Connections among Standards, Clusters or Domains

- Students use proportional reasoning when they analyze scale drawings (7.G.1).
- Students use proportional reasoning and percentages when they extrapolate from random samples and use probability (7.SP.6, 8).

Examples of Opportunities for In-Depth Focus

7.RP.2 Students in grade 7 grow in their ability to recognize, represent and analyze proportional relationships in various ways, including by using tables, graphs and equations.

7.NS.3 When students work toward meeting this standard (which is closely connected to 7.NS.1 and 7.NS.2), they consolidate their skill and understanding of addition, subtraction, multiplication and division of rational numbers.

7.EE.3 This is a major capstone standard for arithmetic and its applications.

7.EE.4 Work toward meeting this standard builds on the work that led to meeting 6.EE.7 and prepares students for the work that will lead to meeting 8.EE.7.

7.G.6 Work toward meeting this standard draws together grades 3–6 work with geometric measurement.

Examples of Opportunities for Connecting Mathematical Content and Mathematical Practices

Mathematical practices should be evident *throughout* mathematics instruction and connected to all of the content areas highlighted above, as well as all other content areas addressed at this grade level. Mathematical tasks (short, long, scaffolded and unscaffolded) are an important opportunity to connect content and practices. Some brief examples of how the content of this grade might be connected to the practices follow.

- When students compare arithmetic and algebraic solutions to the same problem (7.EE.4a), they are identifying correspondences between different approaches (MP.1).
- Solving an equation such as $4 = 8(x - 1/2)$ requires students to see and make use of structure (MP.7), temporarily viewing $x - 1/2$ as a single entity.
- When students notice when given geometric conditions determine a unique triangle, more than one triangle or no triangle (7.G.2), they have an opportunity to construct viable arguments and critique the reasoning of others (MP.3). Such problems also present opportunities for using appropriate tools strategically (MP.5).
- Proportional relationships present opportunities for modeling (MP.4). For example, the number of people who live in an apartment building might be taken as proportional to the number of stories in the building for modeling purposes.

Content Emphases by Cluster[18]

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Key: ■ Major Clusters; ◘ Supporting Clusters; ○ Additional Clusters

Ratios and Proportional Reasoning

■ **Analyze proportional relationships and use them to solve real-world and mathematical problems.**

The Number System

■ **Apply and extend previous understandings of operations with fractions to add, subtract, multiply and divide rational numbers.**

Expressions and Equations

■ **Use properties of operations to generate equivalent expressions.**

■ **Solve real-life and mathematical problems using numerical and algebraic expressions and equations.**

Geometry

○ **Draw, construct and describe geometrical figures and describe the relationships between them.**

○ **Solve real-life and mathematical problems involving angle measure, area, surface area and volume.**

Statistics and Probability

◘ **Use random sampling to draw inferences about a population.**

○ **Draw informal comparative inferences about two populations.**

◘ **Investigate chance processes and develop, use, and evaluate probability models.**

Examples of Linking Supporting Clusters to the Major Work of the Grade

- Use random sampling to draw inferences about a population: The standards in this cluster represent opportunities to apply percentages and proportional reasoning. To make inferences about a population, one needs to apply such reasoning to the sample and the entire population.

- Investigate chance processes and develop, use and evaluate probability models: Probability models draw on proportional reasoning and should be connected to the major work in those standards.

Examples of Key Advances from Grade 7 to Grade 8

- Students build on previous work with proportional relationships, unit rates and graphing to connect these ideas and understand that the points (x, y) on a nonvertical line are the solutions of the equation $y = mx + b$, where m is the slope of the line as well as the unit rate of a proportional relationship (in the case $b = 0$). Students also formalize their previous work with linear relationships by working with functions — rules that assign to each input exactly one output.

- By working with equations such as $x^2 = 2$ and in geometric contexts such as the Pythagorean theorem, students enlarge their concept of number beyond the system of rationals to include irrational numbers. They represent these numbers with radical expressions and approximate these numbers with rationals.

Fluency Expectations or Examples of Culminating Standards

8.EE.7 Students have been working informally with one-variable linear equations since as early as kindergarten. This important line of development culminates in grade 8 with the solution of general one-variable linear equations, including cases with infinitely many solutions or no solutions as well as cases requiring algebraic manipulation using properties of operations. Coefficients and constants in these equations may be any rational numbers.

8.G.9 When students learn to solve problems involving volumes of cones, cylinders and spheres — together with their previous grade 7 work in angle measure, area, surface area and volume (7.G.4–6) — they will have acquired a well-developed set of geometric measurement skills. These skills, along with proportional reasoning (7.RP) and multistep numerical problem solving (7.EE.3), can be combined and used in flexible ways as part of modeling during high school — not to mention after high school for college and careers.[19]

Examples of Major Within-Grade Dependencies

- An important development takes place in grade 8 when students make connections between proportional relationships, lines and linear equations (8.EE, **second cluster**). Making these connections depends on prior grades' work, including 7.RP.2 and 6.EE.9. There is also a major dependency within grade 8 itself: The angle-angle criterion for triangle similarity underlies the fact that a nonvertical line in the coordinate plane has equation $y = mx + b$.[20] Therefore, students must do work with congruence and similarity (8.G.1–5) before they are able to justify

[19] See "Appendix A: Lasting Achievements in K–8."
[20] See page 12 of the *Progression for Expressions and Equations:* http://commoncoretools.files.wordpress.com/2011/04/ccss_progression_ee_2011_04_25.pdf.

the connections among proportional relationships, lines and linear equations. Hence the indicated geometry work should likely begin at or near the very start of the year.[21]

- Much of the work of grade 8 involves lines, linear equations and linear functions (8.EE.5–8; 8.F.3–4; 8.SP.2–3). Irrational numbers, radicals, the Pythagorean theorem and volume (8.NS.1–2; 8.EE.2; 8.G.6–9) are nonlinear in nature. Curriculum developers might choose to address linear and nonlinear bodies of content somewhat separately. An exception, however, might be that when addressing functions, pervasively treating linear functions as separate from nonlinear functions might obscure the concept of function *per se*. There should also be sufficient treatment of nonlinear functions to avoid giving students the misleading impression that all functional relationships are linear (see also 7.RP.2a).

Examples of Opportunities for Connections among Standards, Clusters or Domains

- Students' work with proportional relationships, lines, linear equations and linear functions can be enhanced by working with scatter plots and linear models of association in bivariate measurement data (8.SP.1–3).

Examples of Opportunities for In-Depth Focus

8.EE.5 When students work toward meeting this standard, they build on grades 6–7 work with proportions and position themselves for grade 8 work with functions and the equation of a line.

8.EE.7 This is a culminating standard for solving one-variable linear equations.

8.EE.8 When students work toward meeting this standard, they build on what they know about two-variable linear equations, and they enlarge the varieties of real-world and mathematical problems they can solve.

8.F.2 Work toward meeting this standard repositions previous work with tables and graphs in the new context of input/output rules.

8.G.7 The Pythagorean theorem is useful in practical problems, relates to grade-level work in irrational numbers and plays an important role mathematically in coordinate geometry in high school.

Examples of Opportunities for Connecting Mathematical Content and Mathematical Practices

Mathematical practices should be evident *throughout* mathematics instruction and connected to all of the content areas highlighted above, as well as all other content areas addressed at this grade level.

[21] Note that the Geometry cluster "Understand congruence and similarity using physical models, transparencies or geometry software" supports high school work with congruent triangles and congruent figures.

Mathematical tasks (short, long, scaffolded and unscaffolded) are an important opportunity to connect content and practices. Some brief examples of how the content of this grade might be connected to the practices follow.

- When students convert a fraction such as $^1/_7$ to a decimal, they might notice that they are repeating the same calculations and conclude that the decimal repeats. Similarly, by repeatedly checking whether points are on a line through (1, 2) with slope 3, students might abstract the equation of the line in the form $^{(y\,-\,2)}/_{(x\,-\,1)} = 3$. In both examples, students look for and express regularity in repeated reasoning (MP.8).
- The Pythagorean theorem can provide opportunities for students to construct viable arguments and critique the reasoning of others (e.g., if a student in the class seems to be confusing the theorem with its converse) (MP.3).
- Solving an equation such as $3(x - {}^1/_2) = x + 2$ requires students to see and make use of structure (MP.7).
- Much of the mathematics in grade 8 lends itself to modeling (MP.4). For example, standard 8.F.4 involves modeling linear relationships with functions.
- Scientific notation (8.EE.4) presents opportunities for strategically using appropriate tools (MP.5). For example, the computation $(1.73 \times 10^{-4}) \cdot (1.73 \times 10^{-5})$ can be done quickly with a calculator by squaring 1.73 and then using properties of exponents to determine the exponent of the product by inspection.

Content Emphases by Cluster[22]

Not all of the content in a given grade is emphasized equally in the standards. Some clusters require greater emphasis than the others based on the depth of the ideas, the time that they take to master, and/or their importance to future mathematics or the demands of college and career readiness. In addition, an intense focus on the most critical material at each grade allows depth in learning, which is carried out through the Standards for Mathematical Practice.

To say that some things have greater emphasis is not to say that anything in the standards can safely be neglected in instruction. Neglecting material will leave gaps in student skill and understanding and may leave students unprepared for the challenges of a later grade. All standards figure in a mathematical education and will therefore be eligible for inclusion on the PARCC assessment. However, the assessments will strongly focus where the standards strongly focus.

In addition to identifying the Major, Additional and Supporting Clusters for each grade, suggestions are given following the table on the next page for ways to connect the Supporting to the Major Clusters of the grade. Thus, rather than suggesting even inadvertently that some material not be taught, there is direct advice for teaching it, in ways that foster greater focus and coherence.

[22] Refer to pages 12–14 for further explanation of the cluster-level emphases. Refer also to the Common Core State Standards for Mathematics for the standards that fall within each cluster.

Key: ■ Major Clusters; ◘ Supporting Clusters; ○ Additional Clusters

The Number System	
◘	**Know that there are numbers that are not rational, and approximate them by rational numbers.**
Expressions and Equations	
■	**Work with radicals and integer exponents.**
■	**Understand the connections between proportional relationships, lines and linear equations.**
■	**Analyze and solve linear equations and pairs of simultaneous linear equations.**
Functions	
■	**Define, evaluate and compare functions.**
◘	**Use functions to model relationships between quantities.**
Geometry	
■	**Understand congruence and similarity using physical models, transparencies or geometry software.**
■	**Understand and apply the Pythagorean Theorem.**
○	**Solve real-world and mathematical problems involving volume of cylinders, cones and spheres.**
Statistics and Probability	
◘	**Investigate patterns of association in bivariate data.**

Examples of Linking Supporting Clusters to the Major Work of the Grade

- Know that there are numbers that are not rational, and approximate them by rational numbers: Work with the number system in this grade (8.NS.1–2) is intimately related to work with radicals (8.EE.2), and both of these may be connected to the Pythagorean theorem (8.G, **second cluster**) as well as to volume problems (8.G.9), e.g., in which a cube has known volume but unknown edge lengths.

- Use functions to model relationships between quantities: The work in this cluster involves functions for modeling linear relationships and rate of change/initial value, which supports work with proportional relationships and setting up linear equations.

- Investigate patterns of association in bivariate data: Looking for patterns in scatterplots and using linear models to describe data are directly connected to the work in the Expressions and Equations clusters. Together, these represent a connection to the Standard for Mathematical Practice, MP.4: Model with mathematics.

INTRODUCTION TO THE HIGH SCHOOL STANDARDS ANALYSIS

The Standards for Mathematical Practice are common to both high school and grades K–8, but the Standards for Mathematical Content are organized differently in high school than in grades K–8. In grades K–8, the content standards are organized in a yearly sequence. In high school, the content standards are organized not by year but rather by conceptual category (Functions, Algebra, etc.).

The Model Content Frameworks provide an analysis of the high school standards using terms similar to those used for the grades 3–8 standards analyses. This is done by providing insight into the high school standards using possible courses: Algebra I-Geometry-Algebra II and Mathematics I-Mathematics II-Mathematics III. These potential courses provide a broad overview of content emphasis and connections, as well as the role that the Standards for Mathematical Practice could play. They should not be seen as fully formed courses.

Previous drafts of the Model Content Frameworks for mathematics provided little or no detail about high school courses based on the standards. This version adds more detail about possible courses, including suggesting areas of emphasis in the course introductions, the section on key advances and the section on connections to practices for each course, but it does not specify full details of the courses. The Model Content Frameworks provide initial, high-level guidance.[23]

There are two sections to the high school standards analysis:

1. **General analysis** of the high school standards: analysis that bears on all courses and/or is independent of any particular organization of the standards into courses.
2. **Course-specific analysis** of the high school standards: analysis presented with a view toward two possible high school course sequences.

Please note: The reader is advised to have a copy of *Common Core State Standards for Mathematics* available for use in conjunction with this document. The Model Content Frameworks paraphrase the standards and in some cases refer to standards by code only; readers will need to refer to the standards document for exact language.

[23] Note that the courses outlined in the Model Content Frameworks were informed by, but are not identical to, Appendix A of the Common Core State Standards.

Examples of Opportunities for Connections among Standards, Clusters, Domains or Conceptual Categories

- The standards identify a number of connections among conceptual categories.
    - *Connections among Algebra, Functions and Modeling*. Expressions can define functions; determining an output value for a particular input sometimes involves evaluating an expression. Equivalent expressions on the same domain define the same function. Asking when two different functions have the same value for the same input leads to an equation (e.g., for what x does $x^3 = 2x + 5$?); graphing the two functions allows for finding approximate solutions of the equation. Converting a verbal description to an equation, inequality or system of these is an essential skill in modeling. Because functions often describe relationships among quantities, they are frequently used in modeling. Sometimes functions are defined by a recursive process, which can be modeled effectively using a spreadsheet or other technology.
    - *Connections between Geometry and Algebra*. The correspondence between numerical coordinates and geometric points allows methods from algebra to be applied to geometry and vice versa. The solution set of an equation becomes a geometric curve, making visualization a tool for doing and understanding algebra. Geometric shapes can be described by equations, making algebraic manipulation a tool for geometric understanding, modeling and proof. Geometric transformations provide examples of how the notion of function can be used in geometric contexts; conversely, the effect of replacing f(x) by f(x) + k, k f(x), f(kx) and f(x + k) for specific positive and negative values of k can be interpreted geometrically in terms of transformations on the graphs of the functions.
    - *Connections among Statistics, Functions and Modeling*. Functions may be used to describe data; if the data suggest a linear relationship, the relationship can be modeled with a regression line, and its strength and direction can be expressed through a correlation coefficient.
- The standards also identify a number of connections among standards, clusters and domains.
    - *Connections among standards within Algebra and Functions*. A solid understanding of the correspondence between an equation in two variables and its Cartesian graph is the underpinning for the techniques for graphing lines and quadratics, and it helps students understand what is meant by the "graph of a function." Creating equations and building functions helps students interpret these same objects.
    - *Connections among standards within Geometry.* The progression from congruence to area to similarity can be used to put each of these topics on a logical footing: The basic assumptions that congruent figures have the same area and that area is invariant under finite dissection bring coherence to the formulas for calculating areas of polygonal regions. These formulas, along with results such as the fact that triangles with equal bases and

heights have the same area, can be used to prove properties of dilations and similarity. The triangle similarity criteria are necessary to develop the trigonometry of right triangles.

- *Connections among standards within Statistics and Probability and Functions.* Study of linear associations in statistics and probability (S-ID.6c, 7) builds on students' understanding of linear relationships (cf. F-LE.1). Exploration of quadratic relationships in data on two measurement variables (S-ID.6) depends on understanding key features of a quadratic function and being able to interpret them in terms of a context (F-IF.4).

Examples of Opportunities for Connecting Mathematical Content and Mathematical Practices

- When students use algebra and functions to model a situation, the symbolic calculations they use and the conclusions they draw from those calculations are examples of decontextualizing and contextualizing (reasoning abstractly and quantitatively, MP.2). For example, students looking for a general method of comparing two rate plans with different rates and startup costs ($R_1 = ax + b$, $R_2 = cx + d$) might find the crossover point by working symbolically to solve the equation $ax + b = cx + d$, obtaining the formal solution $x_{\text{crossover}} = (d - b)/(a - c)$. Still thinking symbolically, students can notice that the expression for $x_{\text{crossover}}$ is undefined when $a = c$. Returning to the context, students can see that this makes sense: Two rate plans with the same rate never cross; the better plan in this case is always the one with the lower startup cost. Returning again to the symbolic equation, students can see that in the case of equal rates ($a = c$), the equation for the crossover point reduces to $b = d$, an equation that is true for all x if and only if the two plans have the same startup cost … in which case they are the same plan.

- When students transform expressions purposefully, they are looking for and making use of structure (MP.7).

- When modeling a situation, students often can get started by working repetitively with numerical examples and then look for and express regularity in that repeated reasoning by writing equations or functions (MP.8).

- Throughout high school, students construct viable arguments and critique the reasoning of others (MP.3). As in geometry, important questions in advanced algebra cannot be answered definitively by checking evidence. Results about all objects of a certain type — the factor theorem for polynomials, for example — require general arguments. And deciding whether two functions are equal on an infinite set cannot be settled by looking at tables or graphs; it requires a deeper argument.

- Capturing a situation with precise language (MP.6) can be a critical step toward modeling that situation mathematically. For example, when investigating loan payments, if students can articulate something like, "What you owe at the end of a month is what you owed at the start of the month, plus $1/12$ of the yearly interest on that amount, minus the monthly payment," they are well along a path that will let them construct a recursively defined function for calculating loan payments.

- There are many opportunities in high school to use appropriate tools strategically (MP.5). For example:

  - Students might use graphing calculators or software to gain understanding of the important fact that the graph of an equation in two variables often forms a curve (which could be a line) (A-REI.10). Students might also use graphing calculators and/or graphing software to gain understanding of the important technique of looking for solutions to equations of the form f(x) = g(x) by graphing the solutions of the equations y = f(x) and y = g(x) in the coordinate plane and looking for intersections of the graphs (A-REI.11).

  - Students might use graphing calculators or software to experiment with cases of replacing a function f(x) by f(x) + k, k f(x), f(kx) and f(x + k) for specific positive and negative values of k (F-BF.3).

  - Students might use spreadsheets or similar technology in modeling situations to compute and display recursively defined functions (e.g., a function that gives the balance B_n on a credit card after *n* months given the interest rate, starting balance and regular monthly payment) (F-BF.1a; F-LE).

  - Students might use a computer algebra system to transform or experiment with algebraic expressions (A-APR.6).

  - When making mathematical models, technology is valuable for varying assumptions, exploring consequences and comparing predictions with data (Common Core State Standards, page 72).

  - Technology is usually necessary to work effectively with large data sets or with simulations having many iterations.

  - As students progress in mathematics, they learn techniques that are valuable in a variety of settings. For example, the quadratic formula is a tool in the student's toolkit once it has ceased to become the target of instruction in itself. From then on, it is readily available to the student for use in applications or in reasoning about quadratic equations.

Examples of Content Standards that Apply to Two or More High School Courses

In the high school Standards for Mathematical Content, there are a number of individual content standards or clusters of standards that specify enduring understandings or recurrent themes, such as Seeing Structure in Expressions (A-SSE). Such standards have relevance throughout high school; they are not well thought of as "belonging" to any single high school course. In the charts that follow, several such standards are identified, and a few observations are made about how each standard particularly relates to two or more high school courses in both course sequences.

The standards:

A-REI.4: Solve quadratic equations in one variable.

a. Use the method of completing the square to transform any quadratic equation in *x* into an equation of the form $(x - p)^2 = q$ that has the same solutions. Derive the quadratic formula from this form.

b. Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as *a* ± *bi* for real numbers *a* and *b*.

A-APR: Understand the relationship between zeros and factors of polynomials.

Algebra I (and Mathematics II)	Geometry (and Mathematics II)	Algebra II (and Mathematics III)
Students develop an array of techniques for solving quadratic equations, and they reason about the connections among them, which are based on the idea that different forms of an expression or equation facilitate seeing different features (A-SSE.3). They learn the method of completing the square, interpret it geometrically and use it to derive the quadratic formula, the method that is both general and efficient. Of the techniques for solving quadratic equations, factoring is the most versatile for solving polynomial equations. Factoring depends on the "zero product property" of the real numbers: If a product is 0, at least one of the factors must be 0. In this course, the technique of solving polynomial equations via factoring is applied to quadratic equations. But the principle is perfectly general. Thus, the connection between the linear factors of a polynomial and the zeros of the corresponding polynomial function is a theme that is central to the coherence of high school mathematics. And even in Algebra I, students can solve higher degree equations if they are given a head start on factorizations. For example, they can solve $\mathbf{(x-6)(x^2-5x+6)=0}$ Note: Complex solutions are not emphasized in this course.	While these standards will not generally be emphasized in a Geometry course, completing the square arises again in equations of circles (G-GPE.1). Also, the factorization of a polynomial can reveal structural properties in geometric contexts. An example of this interplay comes from the problem of maximizing the area of a rectangle given a fixed perimeter. One approach to showing that a square does the job is to show how an $a \times b$ rectangle can be dissected to fit inside a square of side-length $\frac{a+b}{2}$. Abstracting from numerical examples, students are led to the algebraic identity $\left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2 = ab$ This can be established by a variety of methods, including the factorization of the left-hand side as a difference of squares. This identity shows how far off a given rectangle is from the corresponding square, and it shows precisely when the "error" $\frac{a-b}{2}$ is 0. The identity can even be used to establish the arithmetic-geometric mean inequality: for non-negative real numbers *a* and *b*, $\frac{a+b}{2} \geq \sqrt{ab}$, with equality if and only if *a* = *b*.	Students complete this standard by including in their repertoire the set of complex numbers, and they continue to use the all these techniques when quadratic factors arise in more general contexts of solving polynomial equations graphing polynomial functions. Factoring remains an important technique more broadly. This course introduces the operation of division with remainder for polynomials in one variable. An analysis of division has several applications that are core to advanced algebra. The most important application is to the factor and remainder theorems (A-APR.2). These theorems are the advancement of the study of solving quadratic equations as they apply to polynomials more generally. The factor theorem deepens the connection between factors of polynomials and solutions to equations that is stated in Algebra I. One of its many applications is that it can be used to show that a polynomial of degree *n* has at most *n* roots, and this implies the important result that a polynomial function of degree *n* is completely determined by *n* + 1 points on its graph.

The standards:

A-SSE.2: Use the structure of an expression to identify ways to rewrite it. ***For example, see*** $x^4 - y^4$ ***as*** $(x^2)^2 - (y^2)^2$***, thus recognizing it as a difference of squares that can be factored as*** $(x^2 - y^2)(x^2 + y^2)$***.***

A-SSE.3: Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.

a. Factor a quadratic expression to reveal the zeros of the function it defines.

b. Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines.

c. Use the properties of exponents to transform expressions for exponential functions. ***For example the expression*** 1.15^t ***can be rewritten as*** $(1.15^{1/12})^{12t} = 1.012^{12t}$ ***to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.***

Algebra I

Once students are accustomed to extending the basic rules of arithmetic to algebraic expressions, they can start to develop the knack for transforming expressions meaningfully. Consider the following expressions:

$$\mathbf{3x + 5x}$$
$$\mathbf{3(x-2) + 5(x-2)}$$
$$\mathbf{3f(x) + 5f(x)}$$
$$\mathbf{x(x-2) + 5(x-2)}$$

Simplifying the first expression is a rote skill often called "collecting like terms." But all of these expressions allow use of the distributive property in exactly the same way. Seeing these as examples of the same idea is seeing structure through "chunking." And the last example is helpful grounding for factoring quadratics.

By encouraging this habit, as students gain experience, they will begin to see a difference of squares embedded in $\mathbf{4x^4 - 9y^6}$, will see $\mathbf{2x^2 - 8x + 8}$ as $\mathbf{2(x-2)^2}$ and will see $\mathbf{25x^2 - 10x - 24}$ as $\mathbf{(5x)^2 - 2(5x) - 24}$, a quadratic in 5*x*.

Writing $\mathbf{x^2 - 2x - 24}$ as $\mathbf{(x-1)^2 - 25}$ by completing the square shows that (for all real values of *x*) the expression is always greater than or equal to −25 and that it assumes that value only when $\mathbf{x = 1}$. Factoring it as $\mathbf{(x-6)(x+4)}$ shows that the expression is zero only when $\mathbf{x = 6}$ or $\mathbf{x = -4}$.

Geometry

In geometry, students can see the structure of an expression in the classic geometric demonstrations of the Pythagorean theorem or the geometric justifications of completing the square.

As another example, students can use geometric dissections to interpret and justify area formulas. Doing so for a trapezoid with base lengths b_1 and b_2 and with height h might lead to several different-looking expressions, such as:

$$\frac{h(b_1+b_2)}{2}, \frac{h}{2}(b_1 + b_2), \left(\frac{b_1}{2} + \frac{b_2}{2}\right)h$$

These are algebraically equivalent, as students can show. Conversely, students can generate different forms of an expression and find dissections that lead to them.

Algebra II

The standards for Arithmetic with Polynomials and Rational Expressions (A-APR) can bring deeper insight into many of the ideas introduced in Algebra I about seeing structure in expressions. Examples like the one given in A-SSE.2 can be used to establish function equality. For example, the fact that

$$x^4 - y^4 = (x^2 - y^2)(x^2 + y^2)$$

implies that the two apparently different functions $x \to \cos^4 x - \sin^4 x$ and $x \to \cos^2 x - \sin^2 x$ are in fact the same function.

The chunking technique finds many applications in Algebra II, ranging from factoring higher degree polynomials to establishing trigonometric identities.

Completing the square from Algebra I can be put into a larger landscape. Completing the square and change of variable allows Algebra I students to see any quadratic as a transformation of $y = x^2$. The same idea might allow Algebra II students to reduce any cubic to $y = x^3$, $y = x^3 + x$ or $y = x^3 - x$.

These standards are about more than developing the skill of changing from one prescribed form to another; just as important is to develop the ability to see which transformation will produce something useful, which is part of the thrust of MP.7: Look for and make use of structure.

The standards:

A-SSE.2: Use the structure of an expression to identify ways to rewrite it. ***For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$.***

A-SSE.3: Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.

- **a. Factor a quadratic expression to reveal the zeros of the function it defines.**
- **b. Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines.**
- **c. Use the properties of exponents to transform expressions for exponential functions.** ***For example the expression 1.15^t can be rewritten as $(1.15^{1/12})^{12t} = 1.012^{12t}$ to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.***

Mathematics I

Once students are accustomed to extending the basic rules of arithmetic to algebraic expressions, they can start to develop the knack for transforming expressions meaningfully. Consider the following expressions:

$$\mathbf{3x + 5x}$$
$$\mathbf{3(x-2) + 5(x-2)}$$
$$\mathbf{3f(x) + 5f(x)}$$
$$\mathbf{x(x-2) + 5(x-2)}$$

Simplifying the first expression is a rote skill often called "collecting like terms." But all of these expressions allow use of the distributive property in exactly the same way. Seeing these as examples of the same idea is seeing structure through "chunking." And the last example is helpful grounding for factoring quadratics.

In geometry contexts, students can use geometric dissections to interpret and justify area formulas. Doing so for a trapezoid with base lengths $\boldsymbol{b_1}$ and $\boldsymbol{b_2}$ and with height *h* might lead to several different-looking expressions, such as:

$$\frac{h(b_1+b_2)}{2}, \frac{h}{2}(b_1 + b_2), \left(\frac{b_1}{2} + \frac{b_2}{2}\right) h$$

These are algebraically equivalent, as students can show. Conversely, students can generate different forms of an expression and find dissections that lead to them.

Mathematics II

By encouraging the habit of seeing structure, as students gain experience, they will begin to see a difference of squares embedded in $4x^4 - 9y^6$, will see $2x^2 - 8x + 8$ as $2(x-2)^2$ and will see $25x^2 - 10x - 24$ as $(5x)^2 - 2(5x) - 24$, a quadratic in 5*x*. Writing $x^2 - 2x - 24$ as $(x-1)^2 - 25$ by completing the square shows that (for all real values of *x*) the expression is always greater than or equal to -25 and that it assumes that value only when $x = 1$. Factoring it as $(x-6)(x+4)$ shows that the expression is zero only when $x = 6$ or $x = -4$.

In geometry, students can see the structure of an expression in the classic geometric demonstrations of the Pythagorean theorem or the geometric justifications of completing the square.

Mathematics III

The standards for Arithmetic with Polynomials and Rational Expressions (A-APR) can bring deeper insight into many of the ideas introduced in Algebra I about seeing structure in expressions. Examples like the one given in A-SSE.2 can be used to establish function equality. For example, the fact that

$$x^4 - y^4 = (x^2 - y^2)(x^2 + y^2)$$

implies that the two apparently different functions $x \rightarrow \cos^4 x - \sin^4 x$ and $x \rightarrow \cos^2 x - \sin^2 x$ are in fact the same function.

The chunking technique finds many applications in Algebra II, ranging from factoring higher degree polynomials to establishing trigonometric identities.

Completing the square from Algebra I can be put into a larger landscape. Completing the square and change of variable allows Algebra I students to see any quadratic as a transformation of $y = x^2$.

These standards are about more than developing the skill of changing from one prescribed form to another; just as important is to develop the ability to see which transformation will produce something useful, which is part of the thrust of MP.7: Look for and make use of structure.

The standards:

F-BF.3: Identify the effect on the graph of replacing *f(x)* by *f(x) + k, k f(x), f(kx)*, and *f(x + k)* for specific values of *k* (both positive and negative); find the value of *k* given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.

Mathematics I (and Algebra I)

When students work with linear functions, they generally will not address this standard explicitly. Nonetheless, students can gain further insights into linear functions and pave the way for this standard by constructing tables, drawing graphs by hand and using graphing technology to compare the graph of the line $y = x$ with graphs of the lines $y = x + k$ and $y = kx$ for various values of k. Students can describe the comparisons using language of transformations (e.g., vertical shift or stretch), and more important, they can use tables, graphs and expressions to explain why the transformations occur. They can then see how these two types of transformations together generate the slope-intercept form of a line. And they may contrast the slope-intercept form with the form $y = m(x - h)$ to begin to consider horizontal shifts.

Students may extend these observations, as appropriate, to other functions studied in the course. For example, students may apply their observations to transformations of simple exponential functions, such as 2^x (over the integers), noting the analogous effects of adding a constant or multiplying by a constant.

Note: Questions and explanations about transformations do not need to be expressed with the generality of function notation. The point here is the habit of seeing and explaining the transformations.

Mathematics II (and Algebra I)

As students begin to explore quadratic functions, they construct tables, draw graphs by hand and use graphing technology to compare the graph of x^2 to the graphs of $x^2 + k$, ax^2 and $(x - h)^2$, for various values of k, a and h. They express the comparisons using language of transformations (e.g., vertical shift or stretch) and use tables, graphs and expressions to explain why the transformations occur. They explain, in particular, why the horizontal translation is h to the right (and why the form is typically written with $x - h$ rather than $x + h$).

Students use these three transformations together to motivate the vertex form of a quadratic: $y = a(x - h)^2 + k$, and they explain why the maximum or minimum occurs when $x = h$. They estimate values of k, h and a given the graphs, and they graph quadratic functions by completing the square to put expressions into vertex form (part of F-IF.8a).

In this course, students begin to use function notation to describe these transformations of graphs and extend these observations and explanations to other families of functions considered in the course, such as absolute value, square root and piecewise functions.

Mathematics III (and Algebra II)

While studying various types of functions (F-IF.7a–e), interpreting key features of graphs in modeling settings (F-IF.4), writing function expressions in different forms (F-IF.8, A-SSE.3) and comparing functions represented in different ways (F-IF.9), students describe and explain graphs of functions as transformations of related functions, and they adjust parameters in a family of functions to choose a model that fits reasonably by eye.

Students explore situations in which two different sequences of transformations result in the same function. For example, they interpret $2^{3(x+2)}$ as -2^x shifted to the left 2 units and "shrunk" along the x-axis by a factor of $1/3$. They also interpret it as $-8^x * 2^6$, a function with a larger base (which thus grows more quickly) along with a vertical stretch. They explain symbolically and graphically why these are the same.

Students use technology to explore the effects of parameter changes in trigonometric functions such as $\sin x$, and they note that their previous observations apply, affecting the amplitude, vertical shift and horizontal shift of the function. They explore the form $\sin kx$ to understand changes in the period of the function as a horizontal stretch.

Students explore graphs of polynomials with only even-degree terms to motivate the definition of an even function, $f(-x) = f(x)$, and they explore polynomials with only odd-degree terms to motivate the definition of an odd function, $f(-x) = -f(x)$. They use whether a function is odd or even to predict symmetry in the graph and vice versa.

The standards:

F-IF.3: Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers. ***For example, the Fibonacci sequence is defined recursively by $f(0) = f(1) = 1$, $f(n+1) = f(n) + f(n-1)$ for $n \geq 1$.***

F-BF.2: Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.

F-LE.2: Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table).

Algebra I (and Mathematics I)

Students talk about sequences as functions, using function notation, either recursively or with explicit expressions, and translating between forms. For example, if a sequence is defined by $f(0) = 3$, $f(n + 1) = f(n) + 5$ for $n > 0$, students use graphs, tables and reasoning to determine an explicit expression $f(n) = 5n + 3$, where n is a whole number, for the same function.

Although students recognize sequences as functions with domains that are subsets of the integers and they plot sequences as dots to distinguish from functions with continuous domains, this course need not insist upon proficient use and interpretation of the domain of a function.

To remain consistent with the role of the number and quantity concepts highlighted in this course, it is suggested that the domain for exponential function be constrained to the integers. Much of the comparison between linear functions and exponential functions takes place over whole-number domains, based on graphs, tables and real-world contexts, and with frequent translation between recursive and explicit formulas. This is the heart of F-BF.2 and F-BF.3 but without the terminology ***arithmetic sequence*** and ***geometric sequence.***

Mathematics II

Although these standards are not a focus of this course, students continue to interpret sequences as functions, as appropriate.

Note: Problems involving sequences should provide enough structure so that the sequence is well defined. Without such structure, students may be asked to find ***a rule*** *(rather than* ***the rule****) that agrees with a sequence because there are infinitely many sequences that agree on a finite set of terms.*

Algebra II (and Mathematics III)

Students are now more formal about their use of the word *sequence,* always specifying the domain, whether described as a sequence or as a function. To recognize that the domain is a crucial part of the description of a function, students begin considering the equality of functions: Two functions are equal if and only if they have the same domain as well as the same output value for each input value in the domain. Thus, when a sequence and a function over the real line are given by the same expression, they are not equal as functions.

Students regularly extend and restrict the domains of functions. For example, they ask, “Can a sequence be extended to the non-negative real line?” The question is essentially about “connecting the dots” on the graph, and students can consider whether such extension makes sense in given contexts. (This work paves the way for interpolation and extrapolation in modeling settings.) A key example is extending exponential functions from whole-number (or integers) domains by defining rational exponents so the rules continue to work (N-RN.1) and then assuming that the rules continue to work for real exponents.

In this course, students recognize arithmetic and geometric sequences (and as special cases of linear and exponential functions, respectively), which completes these standards.

Students use their knowledge of sequences to study series, focusing on arithmetic series (and treating the sequence of partial sums as an example of a quadratic function) and on geometric series, as in A-SSE.4.

HIGH SCHOOL STANDARDS ANALYSIS: COURSE-SPECIFIC ANALYSES INTRODUCTION

Each course is introduced with a high-level narrative. This narrative gives a sense of overall course goals. The description is not intended to be exhaustive.

Course-specific analysis is then provided in the following categories:

Examples of Key Advances from Previous Grades or Courses

- This category highlights some of the major steps in the progression of increasing knowledge and skill from year to year. Note that each key advance in mathematical content also corresponds to a widening scope of problems that students can solve. Examples of key advances are highlighted to stress the need for curricula to treat topics in ways that take into account where students have been in previous grades or courses and where they will be going in subsequent courses.

Fluency Recommendations

- The high school standards do not set explicit expectations for fluency, but fluency is important in high school mathematics. For example, fluency in algebra can help students get past the need to manage computational details so that they can observe structure and patterns in problems. Such fluency can also allow for smooth progress beyond the college and career readiness threshold toward readiness for further study/careers in science, technology, engineering and mathematics (STEM) fields. Therefore, this section makes recommendations about fluencies that can serve students well as they learn and apply mathematics. These fluencies are highlighted to stress the need for curricula to provide sufficient supports and opportunities for practice to help students gain fluency. Fluency is not meant to come at the expense of understanding; it is an outcome of a progression of learning and thoughtful practice. Curricula must provide the conceptual building blocks that develop in tandem with skill along the way to fluency.

Discussion of Mathematical Practices in Relation to Course Content

- This category highlights some of the mathematical practices and describes how they play a role in each course. These examples are provided to stress the need to connect content and practices, as required by the standards.

- In addition to the examples provided in each course, the following are some general comments about connecting content and practices:

  - Connecting content and practices happens in the context of ***working on problems.*** The very first Standard for Mathematical Practice is to make sense of problems and persevere in solving them (MP.1).

  - The Standards for Mathematical Practice interact and overlap with each other. ***They are not***

a checklist.

- Modeling with mathematics is a theme in all high school courses. Modeling problems in high school center on problems arising in everyday life, society and the workplace. Such problems may draw upon mathematical content knowledge and skills articulated in the standards prior to or during the current course. (For more information on modeling in high school, see pages 72 and 73 of the *Common Core State Standards in Mathematics*.)

Please Note

- The words *examples* and *opportunities* in the above categories emphasize that the analysis provided in each category is not exhaustive. For example, there are many opportunities to connect mathematical content and practices in every course, there are many opportunities for in-depth focus in every grade, and so on. A comprehensive description of these features of the standards would be hundreds of pages long. ***The analyses given here should be thought of as starting points.***

- Always refer back to the *Common Core State Standards for Mathematics* for exact language about student expectations.

PARCC Model Content Framework for Mathematics for Algebra I

Students in Algebra I fully master linear equations and linear functions, especially the algebra-geometry interplay regarding slope and graphs. Students also work intensively to master quadratic functions, both from an algebraic and formal perspective as well as in the context of modeling. The work that students do with quadratic functions is connected with and reinforces their work in quadratic equations, polynomial arithmetic and seeing structure in expressions. From an applications perspective, quadratic functions provide opportunities for solving problems involving maxima and minima, an important aspect of modeling. Working intensively with linear and quadratic expressions, equations and functions in Algebra I enables students to focus and master this material.

At the same time, however, students in Algebra I encounter general principles and techniques that apply much more generally than in the linear or quadratic case — for example, learning that the graph of an equation in two variables often forms a curve, which could be a line (A-REI.10). Thus, although most of Algebra I focuses on linear and quadratic equations and functions, the course does include concepts that apply more generally and therefore need to be illustrated beyond the linear and quadratic case. Exponential functions may be discussed in this context but studied in depth later in Algebra II.

Within the domain of Statistics and Probability, Algebra I students work with data on a single count or measurement variable as well as data on two categorical and quantitative variables. Connecting their statistical work with their work in algebra and functions, they also interpret linear models.

To summarize, the critical areas in Algebra I include mastery of linear equations and inequalities, formalization and extension of function concepts (including function notation, domain and range, and exploration of many types of functions, including sequences), linear regression models, quadratic and exponential expressions (including rational exponents), and quadratic functions.

The Standards for Mathematical Practice apply throughout the Algebra I course and, when connected meaningfully with the content standards, allow for students to experience mathematics as a coherent, useful and logical subject. Details about the content and practice standards follow in this analysis.

Examples of Key Advances from Grades K–8

- Having already extended arithmetic from whole numbers to fractions (grades 4–6) and from fractions to rational numbers (grade 7), students in grade 8 encountered particular irrational numbers such as $\sqrt{5}$ or π. In Algebra I, students will begin to understand the real number *system*. For more on the extension of number systems, see page 58 of the standards.

- Students in grade 8 worked with integer exponents. In Algebra I, students will extend the properties of exponents to positive real numbers raised to rational powers (N-RN.1, 2).

- Students in middle grades worked with measurement units, including units obtained by multiplying and dividing quantities. In Algebra I, students apply these skills in a more sophisticated fashion to solve problems in which reasoning about units adds insight (N-Q).

- Themes beginning in middle school algebra continue and deepen during high school. As early as grades 6 and 7, students began to use the properties of operations to generate equivalent expressions (6.EE.3, 7.EE.1). By grade 7, they began to recognize that rewriting expressions in different forms could be useful in problem solving (7.EE.2). In Algebra I, these aspects of algebra carry forward as students continue to use properties of operations to rewrite expressions, gaining fluency and engaging in what has been called "mindful manipulation."[24]

- Students in grade 8 extended their prior understanding of proportional relationships to begin working with functions, with an emphasis on linear functions. In Algebra I, students will master linear and quadratic functions. Students encounter other kinds of functions to ensure that general principles are perceived in generality, as well as to enrich the range of quantitative relationships considered in problems.

- Students in grade 8 connected their knowledge about proportional relationships, lines and linear equations (8.EE.5, 6). In Algebra I, students solidify their understanding of the analytic geometry of lines. They understand that in the Cartesian coordinate plane:

  - The graph of any linear equation in two variables is a line.

  - Any line is the graph of a linear equation in two variables.

- As students acquire mathematical tools from their study of algebra and functions, they apply these tools in statistical contexts (e.g., S-ID.6). In a modeling context, they might informally fit a quadratic function to a set of data, graphing the data and the model function on the same coordinate axes. They also draw on skills they first learned in middle school to apply basic statistics and simple probability in a modeling context. For example, they might estimate a measure of center or variation and use it as an input for a rough calculation.

- Algebra I techniques open a huge variety of word problems that can be solved that were previously inaccessible or very complex in grades K–8. This expands problem solving from grades K–8 dramatically.

Discussion of Mathematical Practices in Relation to Course Content

Two overarching practices relevant to Algebra I are:

- **Make sense of problems and persevere in solving them** (MP.1).

- **Model with mathematics** (MP.4).

Indeed, other mathematical practices in Algebra I might be seen as contributing specific elements of these two. The intent of the following set is not to decompose the above mathematical practices into component parts but rather to show how the mathematical practices work together.

[24] See, for example, "Mindful Manipulation," in *Focus in High School Mathematics: Reasoning and Sense Making* (National Council of Teachers of Mathematics, 2009).

- **Reason abstractly and quantitatively** (MP.2). This practice standard refers to one of the hallmarks of algebraic reasoning, the process of decontextualization and contextualization. Much of elementary algebra involves creating abstract algebraic models of problems (A-CED, F-BF) and then transforming the models via algebraic calculations (A-SSE, A-APR, F-IF) to reveal properties of the problems.

- **Use appropriate tools strategically** (MP.5). Spreadsheets, a function modeling language, graphing tools and many other technologies can be used strategically to gain understanding of the ideas expressed by individual content standards and to model with mathematics.

- **Attend to precision** (MP.6). In algebra, the habit of using precise language is not only a mechanism for effective communication but also a tool for understanding and solving problems. Describing an idea precisely (A-CED, A-REI) helps students understand the idea in new ways.

- **Look for and make use of structure** (MP.7). For example, writing $49x^2 + 35x + 6$ as $(7x)^2 + 5(7x) + 6$, a practice many teachers refer to as "chunking,'' highlights the structural similarity between this expression and $z^2 + 5z + 6$, leading to a factorization of the original: $((7x) + 3)((7x) + 2)$ (A-SSE, A-APR).

- **Look for and express regularity in repeated reasoning** (MP.8). Creating equations or functions to model situations is harder for many students than working with the resulting expressions. An effective way to help students develop the skill of describing general relationships is to work through several specific examples and then express what they are doing with algebraic symbolism (A-CED). For example, when comparing two different text messaging plans, many students who can compute the cost for a given number of minutes have a hard time writing general formulas that express the cost of each plan for *any* number of minutes. Constructing these formulas can be facilitated by methodically calculating the cost for several different input values and then expressing the steps in the calculation, first in words and then in algebraic symbols. Once such expressions are obtained, students can find the break-even point for the two plans, graph the total cost against the number of messages sent and make a complete analysis of the two plans.

Fluency Recommendations

A/G Algebra I students become fluent in solving characteristic problems involving the analytic geometry of lines, such as writing down the equation of a line given a point and a slope. Such fluency can support them in solving less routine mathematical problems involving linearity, as well as in modeling linear phenomena (including modeling using systems of linear inequalities in two variables).

A-APR.1 Fluency in adding, subtracting and multiplying polynomials supports students throughout their work in algebra, as well as in their symbolic work with functions. Manipulation can be more mindful when it is fluent.

A-SSE.1b Fluency in transforming expressions and chunking (seeing parts of an expression as a single object) is essential in factoring, completing the square and other mindful algebraic calculations.

PARCC MODEL CONTENT FRAMEWORK FOR MATHEMATICS FOR GEOMETRY

Students bring many geometric experiences with them to high school; in this course, they begin to use more precise definitions and develop careful proofs. Although there are many types of geometry, this course focuses on Euclidean geometry, studied both with and without coordinates. This course begins with an early definition of congruence and similarity with respect to transformations, then moves on through the triangle congruence criteria and other theorems regarding triangles, quadrilaterals and other geometric figures. Students then move on to right triangle trigonometry and the Pythagorean theorem, which they may extend to the Laws of Sines and Cosines (+). An important aspect of the Geometry course is the connection of algebra and geometry when students begin to investigate analytic geometry in the coordinate plane. In addition, students in Geometry work with probability concepts, extending and formalizing their initial work in middle school. They compute probabilities, drawing on area models. Area models for probability can serve to connect this material to the other aims of the course.

To summarize, high school Geometry corresponds closely to the Geometry conceptual category in the high school standards. Thus, the course involves working with congruence (G-CO), similarity (G-SRT), right triangle trigonometry (in G-SRG), geometry of circles (G-C), analytic geometry in the coordinate plane (G-GPE), and geometric measurement (G-GMD) and modeling (G-MG). The Standards for Mathematical Practice apply throughout the Geometry course and, when connected meaningfully with the content standards, allow for students to experience mathematics as a coherent, useful and logical subject. Details about the content and practice standards follow in this analysis.

Examples of Key Advances from Previous Grades or Courses

- Because concepts such as rotation, reflection and translation were treated in the grade 8 standards mostly in the context of hands-on activities, and with an emphasis on geometric intuition, high school Geometry will put equal weight on precise definitions.
- In grades K–8, students worked with a variety of geometric measures (length, area, volume, angle, surface area and circumference). In high school Geometry, students apply these component skills in tandem with others in the course of modeling tasks and other substantial applications (MP.4).
- The skills that students develop in Algebra I around simplifying and transforming square roots will be useful when solving problems that involve distance or area and that make use the Pythagorean theorem.
- In grade 8, students learned the Pythagorean theorem and used it to determine distances in a coordinate system (8.G.6–8). In high school Geometry, students will build on their understanding of distance in coordinate systems and draw on their growing command of algebra to connect equations and graphs of circles (G-GPE.1).

- The algebraic techniques developed in Algebra I can be applied to study analytic geometry. Geometric objects can be analyzed by the algebraic equations that give rise to them. Some basic geometric theorems in the Cartesian plane can be proven using algebra.

Discussion of Mathematical Practices in Relation to Course Content

- **Reason abstractly and quantitatively** (MP.2). Abstraction is used in geometry when, for example, students use a diagram of a specific isosceles triangle as an aid to reason about *all* isosceles triangles (G-CO.9). Quantitative reasoning in geometry involves the real numbers in an essential way: Irrational numbers show up in work with the Pythagorean theorem (G-SRT.8), area formulas often depend (subtly and informally) on passing to the limit and real numbers are an essential part of the definition of dilation (G-SRT.1). The proper use of units can help students understand the effect of dilation on area and perimeter (N-Q.1).

- **Construct viable arguments and critique the reasoning of others** (MP.3). While all of high school mathematics should work to help students see the importance and usefulness of deductive arguments, geometry is an ideal arena for developing the skill of creating and presenting proofs (G-CO.9.10). One reason is that conjectures about geometric phenomena are often about infinitely many cases at once — for example, *every* angle inscribed in a semicircle is a right angle — so that such results cannot be established by checking every case (G-C.2).

- **Use appropriate tools strategically** (MP.5). Dynamic geometry environments can help students look for invariants in a whole class of geometric constructions, and the constructions in such environments can sometimes lead to an idea behind a proof of a conjecture.

- **Attend to precision** (MP.6). Teachers might use the activity of creating definitions as a way to help students see the value of precision. While this is possible in every course, the activity has a particularly visual appeal in geometry. For example, a class can build the definition of *quadrilateral* by starting with a rough idea ("four sides"), gradually refining the idea so that it rules out figures that do not fit the intuitive idea. Another place in geometry where precision is necessary and useful is in the refinement of conjectures so that initial conjectures that are not correct can be salvaged — two angle measures and a side length do not determine a triangle, but a certain configuration of these parts leads to the angle-side-angle theorem (G-CO.8).

- **Look for and make use of structure** (MP.7). Seeing structure in geometric configurations can lead to insights and proofs. This often involves the creation of auxiliary lines not originally part of a given figure. Two classic examples are the construction of a line through a vertex of a triangle parallel to the opposite side as a way to see that the angle measures of a triangle add to 180 degrees and the introduction of a symmetry line in an isosceles triangle to see that the base angles are congruent (G-CO.9.10). Another kind of hidden structure makes use of area as a device to establish results about proportions, such as the important theorem (and its converse) that a line parallel to one side of a triangle divides the other two sides proportionally (G-SRT.4).

Fluency Recommendations

G-SRT.5 Fluency with the triangle congruence and similarity criteria will help students throughout their investigations of triangles, quadrilaterals, circles, parallelism and trigonometric ratios. These criteria are necessary tools in many geometric modeling tasks.

G-GPE.4, 5, 7 Fluency with the use of coordinates to establish geometric results, calculate length and angle, and use geometric representations as a modeling tool are some of the most valuable tools in mathematics and related fields.

G-CO.12 Fluency with the use of construction tools, physical and computational, helps students draft a model of a geometric phenomenon and can lead to conjectures and proofs.

Building on their work in Algebra I with linear and quadratic functions, students in Algebra II expand their repertoire by working with rational and exponential expressions; polynomial, exponential and logarithmic functions; trigonometric functions with real number domain; and sequences and series. Students work closely with the expressions that define the functions and continue to expand and hone their abilities to model situations and solve equations. Exponential functions, trigonometric functions, and sequences and series all provide opportunities for modeling. As students encounter more and more varied mathematical expressions and functions, general principles they encountered in Algebra I remain relevant, unifying the material in the course.

Students in Algebra II continue their work with Statistics and Probability. They explore and investigate the randomness underlying statistical experiments and make inferences and justify conclusions from sample surveys, experiments and observational studies. They also use probability to evaluate the outcomes of more complex situations than they previously encountered in Geometry.

The critical areas in Algebra II include polynomials (including the structural similarities between the system of polynomials and the system of integers) and polynomial equations, unit circle trigonometry, families of functions (the culmination of all of the types of functions that have been studied and the addition of trigonometric and logarithmic functions), and statistical and probabilistic modeling. The Standards for Mathematical Practice apply throughout the Algebra II course and, when connected meaningfully with the content standards, allow for students to experience mathematics as a coherent, useful and logical subject. Details about the content and practice standards follow in this analysis.

Examples of Key Advances from Previous Grades or Courses

- In Algebra I, students added, subtracted and multiplied polynomials. In Algebra II, students divide polynomials with remainder, leading to the factor and remainder theorems. This is the underpinning for much of advanced algebra, including the algebra of rational expressions.

- Themes from middle school algebra continue and deepen during high school. As early as grade 6, students began thinking about solving equations as a process of reasoning (6.EE.5). This perspective continues throughout Algebra I and Algebra II (A-REI).[25] "Reasoned solving" plays a role in Algebra II because the equations students encounter can have extraneous solutions (A-REI.2).

- In Algebra I, students met quadratic equations with no real roots. In Algebra II, they extend the real numbers to complex numbers, and one effect is that they now have a complete theory of quadratic equations: Every quadratic equation with complex coefficients has (counting multiplicities) two roots in the complex numbers.

[25] See, for example, "Reasoned Solving," in *Focus in High School Mathematics: Reasoning and Sense Making* (National Council of Teachers of Mathematics, 2009).

- In grade 8, students learned the Pythagorean theorem and used it to determine distances in a coordinate system (8.G.6–8). In Geometry, students proved theorems using coordinates (G-GPE.4–7). In Algebra II, students will build on their understanding of distance in coordinate systems and draw on their growing command of algebra to connect equations and graphs of conic sections (e.g., G-GPE.1).

- In Geometry, students began trigonometry through a study of right triangles. In Algebra II, they extend the three basic functions to the entire unit circle.

- As students acquire mathematical tools from their study of algebra and functions, they apply these tools in statistical contexts (e.g., S-ID.6). In a modeling context, they might informally fit an exponential function to a set of data, graphing the data and the model function on the same coordinate axes.

Discussion of Mathematical Practices in Relation to Course Content

While all of the mathematical practice standards are important in all three courses, four are especially important in the Algebra II course:

- **Construct viable arguments and critique the reasoning of others** (MP.3). As in geometry, there are central questions in advanced algebra that cannot be answered definitively by checking evidence. There are important results about *all* functions of a certain type — the factor theorem for polynomial functions, for example — and these require general arguments (A-APR.2). Deciding whether two functions are equal on an infinite set cannot be settled by looking at tables or graphs; it requires arguments of a different sort (F-IF.8).

- **Attend to precision** (MP.6). As in the previous two courses, the habit of using precise language is not only a tool for effective communication but also a means for coming to understanding. For example, when investigating loan payments, if students can articulate something like, "What you owe at the end of a month is what you owed at the start of the month, plus $^1/_{12}$ of the yearly interest on that amount, minus the monthly payment," they are well along a path that will let them construct a recursively defined function for calculating loan payments (A-SSE.4).

- **Look for and make use of structure** (MP.7). The structure theme in Algebra I centered on seeing and using the structure of algebraic expressions. This continues in Algebra II, where students delve deeper into transforming expressions in ways that reveal meaning. The example given in the standards — that $x^4 - y^4$ can be seen as the difference of squares — is typical of this practice. This habit of seeing subexpressions as single entities will serve students well in areas such as trigonometry, where, for example, the factorization of $x^4 - y^4$ described above can be used to show that the functions $\cos^4 x - \sin^4 x$ and $\cos^2 x - \sin^2 x$ are, in fact, equal (A-SSE.2).

  In addition, the standards call for attention to the structural similarities between polynomials and integers (A-APR.1). The study of these similarities can be deepened in Algebra II: Like integers, polynomials have a division algorithm, and division of polynomials can be used to understand the factor theorem, transform rational expressions, help solve equations and factor polynomials.

- **Look for and express regularity in repeated reasoning** (MP.8). Algebra II is where students can do a more complete analysis of sequences (F-IF.3), especially arithmetic and geometric sequences, and their associated series. Developing recursive formulas for sequences is facilitated by the practice of abstracting regularity for how you get from one term to the next and then giving a precise description of this process in algebraic symbols (F-BF.2). Technology can be a useful tool here: Most Computer Algebra Systems allow one to model recursive function definitions in notation that is close to standard mathematical notation. And spreadsheets make natural the process of taking successive differences and running totals (MP.5).

  The same thinking — finding and articulating the rhythm in calculations — can help students analyze mortgage payments, and the ability to get a closed form for a geometric series lets them make a complete analysis of this topic. This practice is also a tool for using difference tables to find simple functions that agree with a set of data.

  Algebra II is a course in which students can learn some technical methods for performing algebraic calculations and transformations, but sense-making is still paramount (MP.1). For example, analyzing Heron's formula from geometry lets one connect the zeros of the expression to the degenerate triangles. As in Algebra I, the modeling practice is ubiquitous in Algebra II, enhanced by the inclusion of exponential and logarithmic functions as modeling tools (MP.4). Computer algebra systems provide students with a tool for modeling all kinds of phenomena, experimenting with algebraic objects (e.g., sequences of polynomials), and reducing the computational overhead needed to investigate many classical and useful areas of algebra (MP.5).

Fluency Recommendations

A-APR.6 This standard sets an expectation that students will divide polynomials with remainder by inspection in simple cases. For example, one can view the rational expression $\frac{x+4}{x+3}$ as $\frac{x+4}{x+3} = \frac{(x+3)+1}{x+3} = 1 + \frac{1}{x+3}$.

A-SSE.2 The ability to see structure in expressions and to use this structure to rewrite expressions is a key skill in everything from advanced factoring (e.g., grouping) to summing series to the rewriting of rational expressions to examine the end behavior of the corresponding rational function.

F-IF.3 Fluency in translating between recursive definitions and closed forms is helpful when dealing with many problems involving sequences and series, with applications ranging from fitting functions to tables to problems in finance.

PARCC Model Content Framework for Mathematics for Mathematics I

Students formalize and deepen their knowledge of linear equations and inequalities, creating expressions, equations and inequalities to represent linear relationships and constraints and using these to solve problems with graphical, numeric and algebraic methods while explaining the reasoning behind their solutions. This knowledge is then extended to exponential expressions and equations, moving from a constant rate of growth to a proportional rate of growth. Students create expressions and equations to represent exponential relationships and use them to solve problems approximately and, in simple cases, algebraically.

Students also formalize and deepen their knowledge of linear functions and begin working with exponential functions. Modeling plays a central role in students' development of knowledge with both linear and exponential functions; exploring a range of contexts that can be modeled with linear and exponential relationships will provide both motivation and meaning to their work. Students understand key features of both classes of functions, including their respective rates of growth, and can represent them in a variety of ways. Students interpret key features of these functions in terms of a context and use this understanding to write function rules for exponential and linear functions. Students also relate linear and exponential models to arithmetic and geometric progressions, including writing them recursively — a skill that comes into play when writing functions to model a situation.

Students advance their work with congruence and transformations. They use tools and methods based on transformations and congruence, such as paper folding and use of dynamic geometry environments, to construct geometric objects and demonstrate geometric properties and relationships.

Within the conceptual area of Statistics and Probability, students represent data on a single count or measurement variable in a variety of ways to better understand measures of center and spread. They explore associations between two measurement variables using a scatter plot, and they begin to use linear models to better understand those associations when appropriate. Throughout their work in Mathematics I, students define quantities and use units appropriately as a way to guide their solutions, as well as use and interpret scales in graphs and data displays, attending to appropriate levels of accuracy.

Examples of Key Advances from Grades K–8

- Students build on previous work with solving linear equations and systems of linear equations in two ways: (a) They extend to more formal solution methods, including attending to the structure of linear expressions, and (b) they solve linear inequalities.

- Students formalize their understanding of the definition of a function, particularly their understanding of linear functions, emphasizing the structure of linear expressions. Students also begin to work on exponential functions, comparing them to linear functions.

- Work with congruence and similarity motions that was begun in grades 6–8 progresses. Students also consider sufficient conditions for congruence of triangles.

- Work with the bivariate data and scatter plots in grades 6–8 is extended to working with lines of best fit.

Discussion of Mathematical Practices in Relation to Course Content

- **Modeling with mathematics** (MP.4) should be a particular focus as students see the purpose and meaning for working with linear and exponential equations and functions.
- **Using appropriate tools strategically** (MP.5) is also important as students explore those models in a variety of ways, including with technology. For example, students might be given a set of data points and experiment with graphing a line that fits the data.
- As Mathematics I continues to develop a foundation for more formal reasoning, students should engage in the practice of **constructing viable arguments and critiquing the reasoning of others** (MP.3).

Fluency Recommendations

A/G High school students should become fluent in solving characteristic problems involving the analytic geometry of lines, such as finding the equation of a line given a point and a slope. This fluency can support students in solving less routine mathematical problems involving linearity, as well as in modeling linear phenomena (including modeling using systems of linear inequalities in two variables).

G High school students should become fluent in using geometric transformation to represent the relationships among geometric objects. This fluency provides a powerful tool for visualizing relationships, as well as a foundation for exploring ideas both within geometry (e.g., symmetry) and outside of geometry (e.g., transformations of graphs).

S Students should be able to create a visual representation of a data set that is useful in understanding possible relationships among variables.

PARCC Model Content Framework for Mathematics for Mathematics II

Students extend their work in algebra with a major emphasis on quadratic relationships. They create expressions and equations to represent situations involving quadratic relationships. They recognize the key features of a quadratic expression and rewrite a quadratic expression in various forms to reveal information about a situation involving a quadratic relationship. They solve quadratic equations using a variety of methods, including using a table or graph to approximate solutions and using algebraic techniques, such as factoring and completing the square, to find exact solutions. They derive and use the quadratic formula. Students graphically explore solving a system of a linear equation and a quadratic equation, and they extend their algebraic techniques to the solution.

Students also extend their work in functions to quadratic and other functions. Students write a function rule representing a quadratic relationship and can also represent that relationship using a table of values or a graph. They explore transformations of the graph of a quadratic function, including horizontal and vertical translations and stretches, looking at the correspondence between changes to the formula and their effects on the graph. They understand important features of a quadratic function, transform function rules into forms that reveal those features and interpret those features in terms of a context. They explore the growth rate of a quadratic function and compare it to linear and exponential functions.

In Statistics and Probability, students consider using a broader range of functions to model a relationship between two quantitative variables and assess the fit of the model. They also begin an exploration of probability, including understanding independence and conditional probability and using rules of probability for computing probabilities of compound events.

Students extend their work with congruence to similarity. When solving problems, investigating geometric properties and demonstrating results, they can use synthetic, transformational and coordinate approaches. They use precise definitions of geometric objects to support their reasoning. Students' proofs focus on explaining their reasoning rather than following a particular form.

Examples of Key Advances from Mathematics I

- Students extend their previous work with linear and exponential expressions, equations, systems of equations and inequalities to quadratic relationships.
- A parallel extension occurs from linear and exponential functions to quadratic functions, where students also begin to analyze functions in terms of transformations.
- Building on their work with transformations, students produce increasingly formal arguments about geometric relationships.

Discussion of Mathematical Practices in Relation to Course Content

- **Modeling with mathematics** (MP.4) should be a particular focus as students see the purpose and meaning for working with quadratic equations and functions, including **using appropriate tools strategically** (MP.5).

- As students explore a variety of ways to represent quadratic expressions, they should **look for and make use of structure** (MP.7).

- As their ability to create and use formal mathematical arguments grows, increased emphasis is placed on students' ability to **attend to precision** (MP.6), as well as to **construct viable arguments and critique the reasoning of others** (MP.3).

Fluency Recommendations

F/S Fluency in graphing functions (including linear, quadratic and exponential) and interpreting key features of the graphs in terms of their function rules and a table of value, as well as recognizing a relationship (including a relationship within a data set), fits one of those classes. This forms a critical base for seeing the value and purpose of mathematics, as well as for further study in mathematics.

A-APR.1 Fluency in adding, subtracting and multiplying polynomials supports students throughout their work in algebra, as well as in their symbolic work with functions. Manipulation can be more mindful when it is fluent.

G-SRT.5 Fluency with the triangle congruence and similarity criteria will help students throughout their investigations of triangles, quadrilaterals, circles, parallelism and trigonometric ratios. These criteria are necessary tools in geometric modeling.

PARCC Model Content Framework for Mathematics for Mathematics III

In the conceptual area of Algebra, students extend their consideration of polynomials, understanding them as a system analogous to the integers and using the structure of an expression to rewrite it in useful ways. They understand the relationship between factors and zeros. They also explore rational expressions. Students reason about and solve a wide range of equations, using graphs or tables of values to approximate solutions or using inspection, factoring or other algebraic techniques when appropriate.

Students analyze an increasingly wide range of functions, including polynomial, trigonometric, logarithmic, rational and other functions. They represent these relationships in different ways and compare functions represented in different ways. They explore and compare key features of these families of functions and express function rules in ways that reveal those features. Students understand logarithmic functions as the inverse of exponential functions and can use inverse functions to solve simple equations. As their understanding of modeling grows, students increasingly use unit analysis as a way to understand problems.

In the conceptual area of Geometry, students consider the relationships between two- and three-dimensional objects. They also apply geometric concepts in modeling situations.

In the conceptual area of Statistics and Probability, the focus is on inferential statistics. Students understand the role of randomization in statistics and can make inferences and justify conclusions from surveys, experiments and observational studies.

Examples of Key Advances from Mathematics II

- Students begin to see polynomials as a system that has mathematical coherence, not just as a set of expressions of a specific type. An analogy to the integers can be made (including operations, factoring, etc.). Subsequently, polynomials can be extended to rational expressions, analogous to the rational numbers.

- The understandings that students have developed with linear, exponential and quadratic functions are extended to considering a much broader range of classes of functions.

- In statistics, students begin to look at the role of randomization in statistical design.

Discussion of Mathematical Practices in Relation to Course Content

- **Modeling with mathematics** (MP.4) continues to be a particular focus as students see a broader range of functions, including **using appropriate tools strategically** (MP.5).

- **Constructing viable arguments and critiquing the reasoning of others** (MP.3) continues to be a focus, as does **attention to precision** (MP.6), because students are expected to provide increasingly precise arguments.

- As students continue to explore a range of algebraic expressions, including polynomials, they should **look for and make use of structure** (MP.7).

- Finally, as students solidify the tools they need to continue their study of mathematics, a focus on **making sense of problems and persevering in solving them** (MP.1) is an essential component for their future success.

Fluency Recommendations

A/F Students should look at algebraic manipulation as a meaningful enterprise, in which they seek to understand the structure of an expression or equation and use properties to transform it into forms that provide useful information (e.g., features of a function or solutions to an equation). This perspective will help students continue to usefully apply their mathematical knowledge in a range of situations, whether their continued study leads them toward college or career readiness.

M Seeing mathematics as a tool to model real-world situations should be an underlying perspective in everything students do, including writing algebraic expressions, creating functions, creating geometric models and understanding statistical relationships. This perspective will help students appreciate the importance of mathematics as they continue their study of it.

N-Q In particular, students should recognize that much of mathematics is concerned with understanding quantities and their relationships. They should pick appropriate units for quantities being modeled, using them as a guide to understand a situation, and be attentive to the level of accuracy that is reported in a solution.

F-BF.1.3 In particular, being able to write a rule to represent a relationship between two quantities is essential to continued meaningful use of algebra. Moreover, students should understand the effects of parameter changes and be able to apply them to create a rule modeling the function.

PARCC MODEL CONTENT FRAMEWORKS FOR MATHEMATICS
ADDITIONAL NOTE ON MODELING (MP.4)

Modeling is a conceptual category in high school (pages 72 and 73 of *Common Core State Standards for Mathematics*) as well as a practice standard (MP.4). The practice standard for modeling reads in part as follows:

> *In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another.*

According to this description, numerical or algebraic word problems can be considered modeling tasks when the emphasis is on using mathematics to understand or reason about the context. However, the quoted text also describes an arc across the grades. During middle grades and certainly by high school, tasks with a strong modeling component will have more of the hallmarks that are described on pages 72 and 73 of the standards, such as a need to attend to issues of precision, a need to select relevant variables, engagement in the steps in the modeling cycle or opportunities to use technology.

Appendix A: Lasting Achievements in K–8[26]

Most of the K–8 content standards trace explicit steps A → B → C in a progression. This can sometimes make it seem as if any given standard exists only for the sake of the next one in the progression. There are, however, culminating or capstone standards (sometimes called "pinnacles") – most of them in the middle grades – that remain important far beyond the particular grade level in which they appear. This is signaled in the standards themselves (page 84):

> *The evidence concerning college and career readiness shows clearly that the knowledge, skills, and practices important for readiness include a great deal of mathematics prior to the boundary defined by (+) symbols in these standards. Indeed, some of the highest priority content for college and career readiness comes from grades 6–8. This body of material includes powerfully useful proficiencies such as applying ratio reasoning in real-world and mathematical problems, computing fluently with positive and negative fractions and decimals, and solving real-world and mathematical problems involving angle measure, area, surface area, and volume. Because important standards for college and career readiness are distributed across grades and courses, systems for evaluating college and career readiness should reach as far back in the standards as grades 6–8. It is important to note as well that cut scores or other information generated by assessment systems for college and career readiness should be developed in collaboration with representatives from higher education and workforce development programs, and should be validated by subsequent performance of students in college and the workforce.*

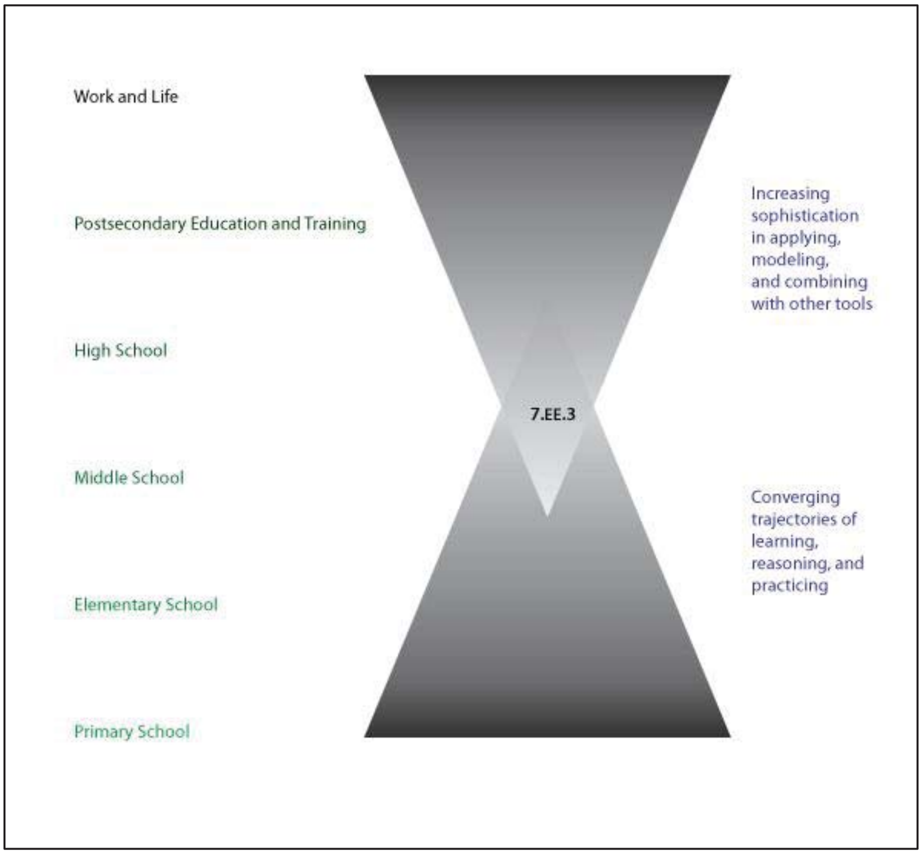
One example of a standard that refers to skills that remain important well beyond middle school is 7.EE.3:

> *Solve multistep real-life and mathematical problems posed with positive and negative rational numbers in any form (whole numbers, fractions, and decimals), using tools strategically. Apply properties of operations to calculate with numbers in any form; convert between forms as appropriate; and assess the reasonableness of answers using mental computation and estimation strategies.* For example: If a woman making $25 an hour gets a 10% raise, she will make an additional 1/10 of her salary an hour, or $2.50, for a new salary of $27.50. If you want to place a towel bar 9 3/4 inches long in the center of a door that is 27 1/2 inches wide, you will need to place the bar about 9 inches from each edge; this estimate can be used as a check on the exact computation.

Other lasting achievements from K–8 would include working with proportional relationships and unit rates (6.RP.3; 7.RP.1, 2); working with percentages (6.RP.3e; 7.RP.3); and working with area, surface area and volume (7.G.4, 6).

As indicated in the quotation from the standards, skills like these are crucial tools for college, work and life. They are not meant to gather dust during high school but are meant to be applied in increasingly flexible ways, for example to meet the high school standards for Modeling. The illustration below shows how these skills fit in with both the learning progressions in the K–8 standards and the demands of the high school standards and readiness for careers and a wide range of college majors.

[26] Excerpted from http://commoncoretools.wordpress.com/2011/06/15/essay-by-jason-zimba-on-pinnacle-standards/.



As shown in the figure, standards like 7.EE.3 are best thought of as descriptions of component skills that will be applied flexibly during high school in tandem with others in the course of modeling tasks and other substantial applications. This aligns with the demands of postsecondary education for careers and for a wide range of college majors. Thus, when students work with these skills in high school, they are not working below grade level, nor are they reviewing. Applying securely held mathematics to open-ended problems and applications is a *higher*-order skill valued by colleges and employers alike.

APPENDIX B: STARTING POINTS FOR TRANSITION TO THE COMMON CORE STATE STANDARDS

Identified in this section are a few particularly rich areas of mathematical content that can be used by assessment designers, teachers, principals, state and district staff members, and teacher educators as starting points to coordinate and concentrate efforts to transition to the Common Core State Standards. Special attention should be given to how well current materials treat these areas. Organizing implementation work according to progressions is recommended because the instructional approach to any given topic should be informed by its place in an overall flow of ideas. Many of these same areas are the focus of the item prototyping currently under way as part of the development of the PARCC Assessment System.

Please note that particular mathematical practices with which to begin are not listed because doing so may unintentionally lead to a misunderstanding of the nature of mathematical practice itself. The mathematical practices are neither a to-do list nor like filing cabinets into which one can sort behaviors. When a student working on a real-world geometry problem in class questions whether another student's drawing is precise enough, the question involves issues of precision as well as modeling, not to mention communication and argument. In short, a single classroom question or behavior might reflect several practices at once.

The following suggestions are not meant to reorganize the standards into a new structure. In fact, a glance will show that the list is incomplete. By providing a focused list of suggested starting points, the risk of taking on too much and doing none of it well is minimized.

- Counting and Cardinality and Operations and Algebraic Thinking (particularly in the development of an understanding of quantity): grades K–2.
- Operations and Algebraic Thinking: multiplication and division in grades 3–5, tracing the evolving meaning of multiplication from equal groups and array/area thinking in grade 3 to all multiplication situations in grade 4 (including multiplicative comparisons) and from whole numbers in grade 3 to decimals and fractions in grades 5 and 6.
- Number and Operations in Base Ten: addition and subtraction in grades 1–4.
- Number and Operations in Base Ten: multiplication and division in grades 3–6.
- Number and Operations – Fractions: fraction addition and subtraction in grades 4–5, including related development of fraction equivalence in grades 3–5.
- Number and Operations – Fractions: fraction multiplication and division in grades 4–6.
- The Number System: grades 6–7.
- Expressions and Equations: grades 6–8, including how this extends prior work in arithmetic.
- Ratio and Proportional Reasoning: its development in grades 6–7, its relationship to functional thinking in grades 6–8, and its connection to lines and linear equations in grade 8.

- Geometry: work with the coordinate plane in grades 5–8, including connections to ratio, proportion, algebra and functions in grades 6–high school.
- Geometry: congruence and similarity of figures in grades 8–high school, with emphasis on real-world and mathematical problems involving scales and connections to ratio and proportion.
- Modeling: focused on equations and inequalities in high school, development from simple modeling tasks such as word problems to richer, more open-ended modeling tasks.
- Seeing Structure in Expressions: from expressions appropriate to grades 8–9 to expressions appropriate to grades 10–11.
- Statistics and Probability: comparing populations and drawing inferences in grades 6–high school.
- Units as a cross-cutting theme in the areas of measurement, geometric measurement, base-ten arithmetic, unit fractions and fraction arithmetic, including the role of the number line.

Many of these stressed areas are likely to be glossed over as "something that is already in the curriculum" — yet the standards require more. ***As noted in the standards, these or any content areas are best approached in the ways envisioned by the Standards for Mathematical Practice.*** The reason for greater focus is to give students and teachers more time — time to discuss, reason with, reflect upon and practice mathematics. These identified areas of mathematics are sufficiently rich to allow the mathematical practices to come alive.

The standards are a challenging vision for higher mathematics performance. By suggesting starting points, the aim is in part to define some content boundaries to help focus the innovation in the creation of new materials and to drive innovation in assessment items.

APPENDIX C: RATIONALE FOR THE GRADES 3–8 CONTENT EMPHASES BY CLUSTER

A close reading of the standards turns up many surface features and concrete details that speak to the standards' emphases. These features and details show some of the ways in which the standards are designed to foster greater focus and coherence in mathematics instruction. For example:

The content domains vary from grade to grade. This is perhaps the most obvious structural feature of the content standards. Traditionally, state standards for mathematics have been organized into content strands that are worded identically for every grade K–8 (if not K–12). By contrast, the content domains in Common Core State Standards vary. This communicates immediately that content emphases shift across the grade bands.

Some content domains are more specific than the traditional content strands.

- Number and Operations in Base Ten (NBT), Number and Operations — Fractions (NF) and The Number System (NS) are all top-level domains. Traditionally, these are often substrands within a larger category such as Number and Operations.

- Expressions and Equations (EE) and Functions (F) are both top-level domains. Traditionally, these have both belonged to a larger category such as Patterns, Functions and Algebra. Ratios and Proportional Relationships is a top-level domain in middle school, whereas this work is usually categorized under Patterns, Functions and Algebra (or under several strands).

Domain names with greater specificity tend to concentrate attention more directly on the priorities of the grade. In the rare case when a state has set priorities, it has been done using framing language; here is a quote from the 2001 *Massachusetts Mathematics Curriculum Framework*:

> *Mathematics in the middle school centers on understanding and computing with rational numbers, and on the study of ratio and proportion (what they are and how they are used to solve problems)*.

This language states with admirable clarity two of the main priorities for middle school. However, those two priorities are not major headings in the framework itself — so it is easy for them to become lost in the list of discrete grade-level requirements. By contrast, the Common Core State Standards make both of these priorities inescapable using the top-level domain structure (NS and RP).

Some domains are **not** more specific than the traditional strands. The grades K–8 domains of Geometry (G), Measurement and Data (MD), and Statistics and Probability (SP) are no more specific than usual. To the extent that greater specificity in top-level categories suggests greater concentration and emphasis, generality in top-level categories suggests comparatively less concentration and emphasis.

Arithmetic accounts for more than three out of five domains in grades K–5. In the traditional picture of content strands, at most two of four strands involve substantial work in arithmetic in early grades: Number and Operations and, to a lesser extent, Algebra. That would tend to suggest that arithmetic in early grades is no more important than the rest of what happens in mathematics in early grades — perhaps even less important, as the Algebra strand traditionally includes a great deal of work outside of arithmetic (e.g., work in extending patterns). But in the early grades of the Common Core State

Standards, three out of five domains are almost wholly concerned with arithmetic.[27] Thus, arithmetic is immediately positioned as a supermajority of instruction at the top level of the content organization.

Work in other K–5 domains also supports arithmetic. Further indications of the strong focus on arithmetic can be seen not only in the obvious domains of Counting and Cardinality (CC), Operations and Algebraic Thinking (OA), NBT and NF but also in other domains. For example, standards relating to area and volume explicitly refer to addition, multiplication and their properties (see 3.MD.7 and 5.MD.5). Also, standards for data representation contain a number of explicit references to major themes in arithmetic. For example, standard 2.MD.10 reads:

> *Draw a picture graph and a bar graph (with single-unit scale) to represent a data set with up to four categories. Solve simple put-together, take-apart, and compare problems using information presented in a bar graph.*

The explicit mention of specific, grade-appropriate word problems is not typical of traditional state standards belonging to the Statistics, Data Analysis and Probability strand. That 2.MD.10 does make such explicit references means that it would be a substantial misinterpretation of this standard to say simply that it is "a standard about picture graphs and bar graphs," as such standards have typically come to be known. Rather, this standard orients picture graphs and bar graphs toward the major work of grade 2. (See Table 1 of the *Progression* for K–3 Categorical Data and 2–5 Measurement Data for further connections of this kind.)

As another example, the word *pattern* first appears in the content standards in grade 3 with standard 3.OA.9:

> *Identify arithmetic patterns (including patterns in the addition table or multiplication table), and explain them using properties of operations.*

The terms *arithmetic*, *addition*, *multiplication* and *properties of operations* do not typically appear in state standards that are "about patterns." That all of these terms do appear in 3.OA.9 makes it a substantial misinterpretation of this standard to say simply that it is "a standard about patterns," as such standards have come to be known. Rather, the standard directs patterns toward the larger purposes of the OA domain. The word *pattern* also appears in the Standards for Mathematical Practice (MP.7), and just as with 3.OA.9, every example given in the practice standard again portrays patterns

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[27] The term *arithmetic* is not being used here to mean computation of sums, differences, products and quotients. That is one important part of arithmetic. But arithmetic in the standards is a large and rich subject that equally involves conceptual understanding, procedural skill and fluency, and problem solving with the basic operations. Moreover, the standards progressions in arithmetic are crafted in such a way as to build a sturdy foundation for algebra in middle school. From the "Progression in Operations and Algebraic Thinking" (May 29, 2011, draft, page 2): "The Progression in Operations and Algebraic Thinking deals with the basic operations—the kinds of quantitative relationships they model and consequently the kinds of problems they can be used to solve as well as their mathematical properties and relationships. Although most of the standards organized under the OA heading involve whole numbers, the importance of the Progression is much more general because it describes concepts, properties, and representations that extend to other number systems, to measures, and to algebra. For example, if the mass of the sun is x kilograms, and the mass of the rest of the solar system is y kilograms, then the mass of the solar system as a whole is the sum $x + y$ kilograms. In this example of additive reasoning, it doesn't matter whether x and y are whole numbers, fractions, decimals, or even variables. Likewise, a property such as distributivity holds for all the number systems that students will study in K–12, including complex numbers."

(and more generally structure) being put to some use, instead of forming a separate object of study that detracts from focus.

Some clusters are explicitly connected to others; some clusters stand more alone. Some clusters in any given grade naturally stand somewhat apart from others. Examples of these would include:

- Many Geometry clusters, such as those relating to hierarchies of shapes, congruence or similar subjects. These are typically connected more weakly to arithmetic clusters than arithmetic clusters are connected to each other.
- The first cluster in 6.SP, "Develop understanding of statistical variability." This introduces into the standards the statistical notions of variability and distribution, center and spread. These are, strictly speaking, not mathematical ideas,[28] so it is natural that they do not connect tightly to, say, applying properties of operations to generate equivalent expressions.
- The Statistics and Probability clusters in 7.SP, which introduce into the standards the notions of randomness, probability, random sampling and comparison of populations.

This is not to say that one might not devise connections to these clusters, if desired; rather it is to say that in other cases, connections are explicit and unavoidable in the standards. For example, 6.EE.9 ties its cluster explicitly to 6.RP; 7.G.1 ties its cluster explicitly to 7.RP; and 8.SP.3 ties its cluster explicitly to 8.F.

A close reading of the *Progressions* also turns up some surface features and concrete details that shed light on some emphases in the standards. Some clusters receive more extensive discussion than others. For example, consider the three clusters in grade 4 OA:

Use the four operations with whole numbers to solve problems.

Gain familiarity with factors and multiples.

Generate and analyze patterns.

Standards are not traditionally written at "uniform grain size" but are often interpreted as such. Some things are quick to state but take a long time in the classroom; others take many words to describe but are simpler to address instructionally. One might have inferred that each of these three clusters was intended to have equal emphasis at grade 4. However, a careful reading of the full body of OA standards dispels this notion. So does even a superficial reading of the *Progression* for OA. There, for instance, we find that:

- Three times as many references are made to standards in the first cluster as are made to standards in the next two clusters combined (6-1-1);

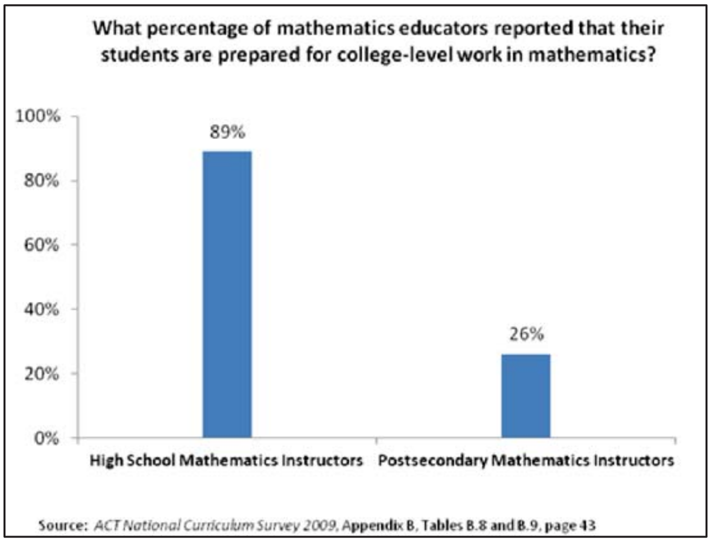
[28] From the Guidelines for Assessment and Introduction in Statistics Education project report, section on "The Difference between Statistics and Mathematics," page 6: "A major objective of statistics education is to help students develop statistical thinking. Statistical thinking, in large part, must deal with this omnipresence of variability; statistical problem solving and decision making depend on understanding, explaining and quantifying the variability in the data. It is this focus on variability in data that sets apart statistics from mathematics."

- Four additional references are made to standards in other domains that connect directly to standards in the first cluster, while no such additional references are made in either of the next two clusters (4-0-0);
- More words are used to explain the standards in the first cluster than are used to explain the standards in the next two clusters combined; and
- Both of the illustrative problems provided in the margin relate directly to the first cluster, while no effort was expended on providing illustrative problems that relate directly to either of the next two clusters.

These visible details — any of which, in principle*,* might have gone the other way — begin to reveal the relative emphases in the standards.

APPENDIX D: CONSIDERATIONS FOR COLLEGE AND CAREER READINESS

In grades K–8, the Model Content Frameworks provide cluster-level emphases to help ensure that implementation efforts preserve the focus and coherence of the standards and that students remain on track to college and career readiness. Cluster-level emphases have not been provided for high school courses because the Model Content Frameworks themselves do not include full details about course-by-course content choices. However, general guidance is provided in this appendix about some of the most important aspects of the standards in relation to college and career readiness.



Surveys have shown repeatedly that high school mathematics instructors and postsecondary mathematics instructors tend to differ in their views about the importance of particular knowledge and skills as prerequisites for success in entry-level, credit-bearing college mathematics courses (ACT 2006, 2009). When postsecondary instructors in these surveys are asked to rate the importance of various mathematics topics to college readiness, they tend to make sharper distinctions than do high school instructors. Postsecondary instructors in these surveys tend to value mastery of fundamentals over broad topic coverage.

This sentiment has been echoed during PARCC's ongoing discussions with higher education stakeholders. In those discussions, postsecondary instructors have stressed the importance of deeper learning of fundamental mathematics. That includes being able to approach problems in the ways described in the Standards for Mathematical Practice. Postsecondary instructors also stressed the importance of being able to solve complex problems using securely held knowledge and skills. The ability to flexibly apply what one already knows to a nonroutine or complex problem is an important aspect of readiness for college and careers.[29] Although PARCC's stakeholder discussions do not themselves have the scientific weight of a well-designed national survey, it is reassuring to see the same themes reinforced in both settings.

Educators in high school can help bridge this gap. To that end, educators can devote particular energy to the following aspects of the standards, which play a prominent role in college and career readiness:

- The Standards for Mathematical Practice, viewed in connection with mathematical content.
- Modeling and rich applications (see pages 72 and 73 in the standards), which can be integrated into mathematics curriculum, instruction and assessment.

[29] See also "Appendix A: Lasting Achievements in K–8."

- Note in particular the standards in high school marked with a star symbol (★). Star symbols identify potential opportunities to integrate content with the modeling practice.
- Note also that modeling is a sophisticated practice; this means that modeling and other complex tasks will naturally draw upon securely held knowledge and skills. Many tasks in high school will demand flexible application of content knowledge first gained in grades 6–8 to solve complex problems. (See page 84 of the standards.)

- The following particular clusters of high school standards, which have wide relevance as prerequisites for a range of postsecondary college and career pathways:
  - Number and Quantity: Quantities:
    - Reason quantitatively and use units to solve problems.
  - Number and Quantity: The Real Number System:
    - Extend the properties of exponents to rational exponents.
    - Use properties of rational and irrational numbers.
  - Algebra: Seeing Structure in Expressions:
    - Interpret the structure of expressions.
    - Write expressions in equivalent forms to solve problems.
  - Algebra: Arithmetic with Polynomials and Rational Expressions:
    - Perform arithmetic operations on polynomials.
  - Algebra: Creating Equations:
    - Create equations that describe numbers or relationships.
  - Algebra: Reasoning with Equations and Inequalities:
    - Understand solving equations as a process of reasoning and explain the reasoning.
    - Solve equations and inequalities in one variable.
    - Represent and solve equations and inequalities graphically.
  - Functions: Interpreting Functions:
    - Understand the concept of a function and use function notation.
    - Analyze functions using different representations.
    - Interpret functions that arise in applications in terms of a context.

- Functions: Building Functions:
    - Build a function that models a relationship between two quantities.
- Geometry: Congruence:
    - Prove geometric theorems.
- Statistics and Probability: Interpreting Categorical and Quantitative Data:
    - Summarize, represent and interpret data on a single count or measurement variable.

PARCC will be creating a portfolio of assessments for high school courses. Student scores from these assessments will contribute to an overall determination of college and career readiness valid for informing postsecondary educational decisions such as college admissions. The overall determination will respond to evidence about college and career readiness by addressing mastery of fundamentals as well as the ability to solve complex problems by applying securely held knowledge and skills.

In the best view, the college- and career-ready line in the standards can be seen as a milepost, not a finish line; it is a line best crossed with velocity. In particular, students who wish to pursue science, technology, engineering or mathematics majors, or who wish to do college-level work in high school such as Advanced Placement or International Baccalaureate courses, must progress well beyond the initial threshold of college and career readiness as defined by the standards.